INTERIOR GRADIENT ESTIMATES FOR QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We study quasilinear elliptic equations of the form $\operatorname{div}\mathbf{A}(x,u,\nabla u) = \operatorname{div}\mathbf{F}$ in bounded domains in $\mathbb{R}^n$, $n \geq 1$. The vector field $\mathbf{A}$ is allowed to be discontinuous in x, Lipschitz continuous in u and its growth in the gradient variable is like the p-Laplace operator with $1 < p < \infty$. We establish interior $W^{1,q}$-estimates for locally bounded weak solutions to the equations for every $q > p$, and we show that similar results also hold true in the setting of *Orlicz* spaces. Our regularity estimates extend results which are only known for the case $\mathbf{A}$ is independent of u and they complement the well-known interior $C^{1,\alpha}$- estimates obtained by DiBenedetto [9] and Tolksdorf [33] for general quasilinear elliptic equations.

1. Introduction

We will investigate interior regularity for weak solutions to degenerate quasilinear elliptic equations of the form

$$\operatorname{div}\mathbf{A}(x,u,\nabla u) = \operatorname{div}\mathbf{F} \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$. Without loss of generality we take Ω to be the Euclidean ball $B_6 := \{x \in \mathbb{R}^n : |x| < 6\}$. Let $\mathbb{K} \subset \mathbb{R}$ be an open interval and consider general vector field

$$\mathbf{A} = \mathbf{A}(x,z,\xi) : B_6 \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

which is a Carathéodory map, that is, $\mathbf{A}(x,z,\xi)$ is measurable in x for every $(z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n$ and continuous in (z,ξ) for a.e. $x \in B_6$. We assume that there exist constants $\Lambda > 0$ and $1 < p < \infty$ such that $\mathbf{A}$ satisfies the following structural conditions for a.e. $x \in B_6$:

$$\langle \mathbf{A}(x,z,\xi) - \mathbf{A}(x,z,\eta), \xi - \eta\rangle \geq \Lambda^{-1}(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \quad \forall z \in \overline{\mathbb{K}} \text{ and } \forall \xi, \eta \in \mathbb{R}^n, \tag{1.2}$$

$$|\mathbf{A}(x,z,\xi)| \leq \Lambda|\xi|^{p-1} \qquad \forall (z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n, \tag{1.3}$$

$$|\mathbf{A}(x,z_1,\xi) - \mathbf{A}(x,z_2,\xi)| \leq \Lambda|\xi|^{p-1}|z_1 - z_2| \qquad \forall z_1, z_2 \in \overline{\mathbb{K}} \text{ and } \forall \xi \in \mathbb{R}^n. \tag{1.4}$$

We want to emphasize that (1.2)–(1.4) are required to hold only for $z \in \overline{\mathbb{K}}$. This is useful since in some applications, (1.2)–(1.4) are satisfied only when $\mathbb{K}$ is a strict subset of $\mathbb{R}$ (see [17] for such an example where $\mathbb{K} = (0, M_0)$ for some constant $M_0 > 0$).

The class of equations of the form (1.1) with $\mathbf{A}$ satisfying (1.2)–(1.4) contains the well-known p-Laplace equations. The interior $C^{1,\alpha}$ regularity for homogeneous p-Laplace equations was established by Uralćeva [35], Uhlenbeck [34], Evans [14] and Lewis [24], while interior $W^{1,q}$-estimates for nonhomogeneous p-Laplace equations were obtained by Iwaniec [19] and DiBenedetto and Manfredi [10]. More generally, (1.1) includes equations of the type

$$\operatorname{div}\mathbf{A}(x,\nabla u) = \operatorname{div}\mathbf{F} \quad \text{in} \quad \Omega \tag{1.5}$$

whose $W^{1,q}$ regularity has been studied by several authors when $\mathbf{A}$ is not necessarily continuous in the x variable [2, 3, 6, 11–13, 19, 20, 28, 29, 31].

In this paper we study general quasilinear equations (1.1) when the principal parts also depend on the z variable. In the case $\mathbf{A}$ is Lipschitz continuous in both x and z variables, the interior $C^{1,\alpha}$ regularity for locally bounded weak solutions to the corresponding homogeneous equations was established by DiBenedetto [9] and Tolksdorf [33] (see also [25] and the books [16, 22, 27] for

further results). When $\mathbf{A}$ is discontinuous in x, one does not expect Hölder estimates for gradients of weak solutions and it is natural to search for L^q- estimates for the gradients instead. However, this type of estimates for solutions to (1.1) is not well understood even if $\mathbf{F} = 0$. Our main purpose of the current work is to address this issue by establishing $W^{1,q}$-estimates for locally bounded weak solutions to the nonhomogeneous equation (1.1) when $\mathbf{A}$ is not necessarily continuous in the x variable and $\mathbf{F}$ belongs to the L^q space. We achieve this in Theorem 2.4 whose particular consequence gives the following result:

Theorem 1.1. *Let $\mathbb{K} \subset \mathbb{R}$ be an open interval and $M_0 > 0$. Let $\mathbf{A} : B_6 \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a Carathéodory map such that $(z, \xi) \mapsto \mathbf{A}(x, z, \xi)$ is differentiable on $\mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ for a.e. $x \in B_6$. Assume that $\mathbf{A}$ satisfies* (1.2)–(1.4) *and the following conditions for a.e. $x \in B_6$ and for all $z \in \overline{\mathbb{K}}$:*

$$\langle \partial_\xi \mathbf{A}(x, z, \xi)\eta, \eta\rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2 \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \forall \eta \in \mathbb{R}^n,$$

$$|\partial_\xi \mathbf{A}(x, z, \xi)| \leq \Lambda|\xi|^{p-2} \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then for any $q > p$, there exists a constant $\delta = \delta(p, q, n, \Lambda, \mathbb{K}, M_0) > 0$ such that: if

$$\sup_{0<\rho\leq 3}\sup_{y\in B_1} \fint_{B_\rho(y)} \Big[\sup_{z\in\overline{\mathbb{K}}}\sup_{\xi\neq 0} \frac{|\mathbf{A}(x, z, \xi) - \mathbf{A}_{B_\rho(y)}(z, \xi)|}{|\xi|^{p-1}}\Big] dx \leq \delta, \tag{1.6}$$

and u is a weak solution of

$$\operatorname{div} \mathbf{A}(x, u, \nabla u) = \operatorname{div} \mathbf{F} \quad \text{in} \quad B_6$$

satisfying $\|u\|_{L^\infty(B_5)} \leq M_0$, we have

$$\|\nabla u\|_{L^q(B_1)} \leq C\left(\|u\|_{L^p(B_6)} + \||\mathbf{F}|^{\frac{1}{p-1}}\|_{L^q(B_6)}\right).$$

Here $\mathbf{A}_{B_\rho(y)}(z, \xi) := \fint_{B_\rho(y)} \mathbf{A}(x, z, \xi)\, dx$ and C is a constant depending only on q, p, n, Λ, $\mathbb{K}$ and M_0.

Condition (1.6) means that the BMO modulus of $\mathbf{A}$ in the x variable is sufficiently small and hence it is automatically satisfied when $x \mapsto \mathbf{A}(x, z, \xi)$ is of vanishing mean oscillation. In particular, (1.6) allows $\mathbf{A}$ to be discontinuous in x. We note that some smallness condition in x for $\mathbf{A}$ is necessary since it was known from Meyers' work [30] that in general weak solutions to (1.5) do not possess interior $W^{1,q}$-estimates for every $q > p$ even in the linear case (i.e. $\mathbf{A}(x, \nabla u) = \mathbf{A}(x)\nabla u$ and $p = 2$).

$W^{1,q}$ theory for equation (1.5) was pioneered by Caffarelli and Peral. In [6], these authors derived interior $W^{1,q}$-estimates for solutions to (1.5) when $\mathbf{A}$ is sufficiently close in the L^∞ sense to its average in the x variable in every small scales. For the case $\mathbf{A}(x, \xi) = \langle \mathbf{A}(x)\xi, \xi\rangle^{\frac{p-2}{2}}\mathbf{A}(x)\xi$ with the matrix $\mathbf{A}(x)$ being uniformly elliptic and bounded, Kinnunen and Zhou [20] obtained interior $W^{1,q}$-estimates when $\mathbf{A}(x) \in VMO$, i.e. $\mathbf{A}(x)$ is of vanishing mean oscillation. Recently, Byun and Wang [2] (see also [3]) were able to obtain $W^{1,q}$-estimates for (1.5) under the assumption that the BMO modulus of $\mathbf{A}$ in the x variable is sufficiently small. Our obtained estimates in Theorem 1.1 are the same spirit as [2] but for general quasilinear elliptic equations of the form (1.1).

The proofs of $W^{1,q}$-estimates for solutions to (1.5) in the above mentioned work use the perturbation technique from [4–6] and rely essentially on the central fact that equations of this type are invariant with respect to dilations and rescaling of domains. Unfortunately, this is no longer true for equations of the general form (1.1) and this presents a serious obstacle in deriving $W^{1,q}$-estimates for their solutions. Our idea to handle this issue is to enlarge the class of equations under consideration in a suitable way by considering the associated quasilinear elliptic equations with two parameters (see equation (2.3)). The class of these equations is the smallest one that is invariant with respect to dilations and rescaling of domains and that contains equations of the form (1.1). Given the invariant structure, a key step in our derivation of $W^{1,q}$-estimates for the solution u is to be able to approximate ∇u by a good gradient in L^p norm in a suitable sense (see Corollary 5.2).

However, with the more general class of equations there arise new difficulties in this task as we need to obtain the approximation uniformly with respect to the two parameters. We achieve this in Lemma 4.6 and Corollary 5.2, and it is crucial that the constants δ there are independent of the two parameters λ and θ. The main technical point of this paper is Lemma 4.6 which is a key point in our proof and is obtained through a delicate compactness argument. This kind of compactness arguments with parameters was first introduced in our recent paper [17] where parabolic equations whose principle parts are linear in the gradient variable were considered. Here we extend further the argument to take care the highly nonlinear structure in gradient of our equation (1.1). Enlarging the class of equations to ensure the invariances while still being able to obtain intermediate estimates uniformly with respect to the enlargement is the main reason for our achievement and is the novelty of this work.

Our obtained interior $W^{1,q}$-estimates in Theorem 1.1 for general quasilinear elliptic equations extend the corresponding estimates derived in [2,6] for equation (1.5). These estimates complement the celebrated interior $C^{1,\alpha}$-estimates by DiBenedetto [9] and Tolksdorf [33] for (1.1). In fact, Theorem 1.1 is a particular consequence of our more general result established in Theorem 2.4. It is worth pointing out that Theorem 2.4 is even new when restricted to the simpler equation (1.5). Indeed, we only assume that the distance from $\mathbf{A}(x,\xi)$ to a large set of "good" vector fields to be small while the previous work requires the distance from $\mathbf{A}(x,\xi)$ to its average in the x variable to be small. More importantly, we identify the properties of these good vector fields and are able to implement the general idea that weak solutions to (1.1) possess interior $W^{1,q}$- estimates for any $q \in (p,\infty)$ provided that the equation is sufficiently close to a homogeneous equation of similar form whose Dirichlet problem has a unique weak solution admitting interior $W^{1,\infty}$-estimates.

The method of our proofs in this paper is quite robust and we illustrate this in Subsection 6.2 by showing that the interior estimates obtained in Theorem 2.4 still hold true in the setting of *Orlicz* spaces (see Theorem 6.6 for the precise statement). We end the introduction by noting that quasilinear equations of general structures (1.2)–(1.4) arise in several applications and the availability of $W^{1,q}$-estimates for their solutions might be helpful for answering some open questions in these problems. We refer readers to [17] for such an application of $W^{1,q}$-estimates.

2. Quasilinear elliptic equations of p-Laplacian type and main results

Our goal is to derive interior $W^{1,q}$- estimates for weak solutions to

$$\operatorname{div} \mathbf{A}(x,u,\nabla u) = \operatorname{div} \mathbf{F} \quad \text{in} \quad B_6 \tag{2.1}$$

for any $q \in (p,\infty)$. We shall show that this is possible if (2.1) is close to a homogeneous equation of similar form whose Dirichlet problem has a unique weak solution admitting interior $W^{1,\infty}$-estimates. For this purpose, we introduce in the next subsection the class of reference equations together with a quantity used to measure the closeness between two equations. In Subsection 2.2, we explain the reasons for enlarging the class of equations under consideration.

2.1. **The class of good reference equations.** Let $\eta : \overline{\mathbb{K}} \times [0,\infty) \to \mathbb{R}$ be such that $\lim_{r\to 0^+} \eta(z,r) = \eta(z,0) = 0$ for each $z \in \overline{\mathbb{K}}$. Let $\mathbb{G}_{B_3}(\eta)$ denote the class of all vector fields $\mathbf{a} : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying conditions (1.2)– (1.4) for a.e. $x \in B_3$ together with the following additional properties:

(H1) For a.e. $x \in B_3$ and every $z \in \overline{\mathbb{K}}$, the map $\xi \mapsto \mathbf{a}(x,z,\xi)$ is continuously differentiable away from the origin with

$$|\partial_\xi \mathbf{a}(x,z,\xi)| \leq \Lambda |\xi|^{p-2} \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

(H2) For every $z \in \overline{\mathbb{K}}$, we have

$$\sup_{\xi \neq 0} \sup_{x_0:\, B_r(x_0) \subset B_3} \fint_{B_r(x_0)} \frac{\left|\mathbf{a}(x,z,\xi) - \mathbf{a}_{B_r(x_0)}(z,\xi)\right|}{|\xi|^{p-1}}\, dx \leq \eta(z,r) \quad \text{for all } r > 0 \text{ small.}$$

(H3) For any $M > 0$ and $0 < R \leq 1$, if $\overline{v}$ is a weak solution to $\operatorname{div} \mathbf{a}(x, \overline{v}, \nabla \overline{v}) = 0$ in B_{3R} satisfying $\|\overline{v}\|_{L^\infty(B_{3R})} \leq M$ then we have

$$\|\nabla \overline{v}\|^p_{L^\infty(B_{\frac{5r}{6}})} \leq C(p,n,\eta,\Lambda,\mathbb{K},M) \fint_{B_r} |\nabla \overline{v}|^p\, dx \quad \forall 0 < r \leq 3R.$$

By taking $\mathbf{a}(x,z,\xi) = \mathbf{a}(\xi)$, it is clear that the class $\mathbb{G}_{B_3}(\eta)$ is nonempty. In fact, it contains a large number of vector fields as shown in our recent paper [18]. Ones also find in [18] that the class of vector fields considered in [9, 33] to derive interior $C^{1,\alpha}$- estimates for the corresponding homogeneous equations belongs to $\mathbb{G}_{B_3}(\eta)$ for $\eta(z,r) \equiv \gamma_1 r$ with γ_1 being some positive constant.

Definition 2.1. *Let $y \in B_1$, $0 < \rho \leq 3$, and $B_\rho(y) := \{x \in \mathbb{R}^n : |x - y| < \rho\}$.*

(i) *We define*

$$\mathbb{G}_{B_\rho(y)}(\eta) := \left\{ \mathbf{a}(x,z,\xi) := \mathbf{a}'\Big(\frac{3(x-y)}{\rho}, z, \xi\Big) \ \textit{for}\ (x,z,\xi) \in B_\rho(y) \times \overline{\mathbb{K}} \times \mathbb{R}^n \,\Big|\, \mathbf{a}' \in \mathbb{G}_{B_3}(\eta) \right\}.$$

(ii) *Let $\mathbf{A} : B_\rho(y) \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a Carathéodory map. Define the distance*

$$dist\Big(\mathbf{A}, \mathbb{G}_{B_\rho(y)}(\eta)\Big) := \inf_{\mathbf{a} \in \mathbb{G}_{B_\rho(y)}(\eta)} \fint_{B_\rho(y)} \Big[\sup_{z \in \overline{\mathbb{K}}} \sup_{\xi \neq 0} \frac{|\mathbf{A}(x,z,\xi) - \mathbf{a}(x,z,\xi)|}{|\xi|^{p-1}}\Big]\, dx.$$

Remark 2.2. *We note that under conditions* (1.2)–(1.4) *for the vector field* $\mathbf{a}$*, the homogeneous equation div* $\mathbf{a}(x,v,\nabla v) = 0$ *in* B_3 *admits the comparison principle (see [8, Theorem 1.2]). This together with the classical existence result due to Leray and Lions [23,26] (see also [7, Theorem 2.8]) ensures that: for any $u \in W^{1,p}(B_3)$ with $u(x) \in \overline{\mathbb{K}}$ for a.e. $x \in B_3$, the Dirichlet problem*

$$\begin{cases} div\, \mathbf{a}(x,v,\nabla v) &= 0 \quad in \quad B_3, \\ v &= u \quad on \quad \partial B_3 \end{cases}$$

has a unique weak solution $v \in W^{1,p}(B_3)$ satisfying $v(x) \in \overline{\mathbb{K}}$ for a.e. $x \in B_3$.

2.2. **Quasilinear equations with two parameters.** Let us consider a function $u \in W^{1,p}_{loc}(B_{rR})$ such that $u(y) \in \overline{\mathbb{K}}$ for a.e. $y \in B_{rR}$ and u satisfies

$$\operatorname{div} \mathbf{A}(y,u,\nabla u) = \operatorname{div} \mathbf{F} \quad \text{in} \quad B_{rR}$$

in the sense of distribution. Then the rescaled function

$$v(x) := \frac{u(rx)}{\mu\, r} \quad \text{for} \quad r,\, \mu > 0 \tag{2.2}$$

has the properties: $v(x) \in \frac{1}{\mu r}\overline{\mathbb{K}}$ for a.e. $x \in B_R$ and v solves the equation

$$\operatorname{div} \mathbf{A}_{\mu,r}(x, \mu r v, \nabla v) = \operatorname{div} \mathbf{F}_{\mu,r} \quad \text{in} \quad B_R$$

in the distributional sense. Here,

$$\mathbf{A}_{\mu,r}(x,z,\xi) := \frac{\mathbf{A}(rx,z,\mu\xi)}{\mu^{p-1}} \quad \text{and} \quad \mathbf{F}_{\mu,r}(x) := \frac{\mathbf{F}(rx)}{\mu^{p-1}}.$$

It is clear that if $\mathbf{A} : B_{rR} \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies conditions (1.2)–(1.4), then the rescaled vector field $\mathbf{A}_{\mu,r} : B_R \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ also satisfies the structural conditions with the same constants.

The above observation shows that equations of type (2.1) are not invariant with respect to the standard scalings (2.2). This presents a serious obstacle in obtaining $W^{1,q}$-estimates for their solutions as they do not generate enough estimates to carry out the proof by using existing methods. Our idea is to enlarge the class by considering associated quasilinear equations with two parameters

$$\operatorname{div}\Big[\frac{\mathbf{A}(x,\lambda\theta u,\lambda\nabla u)}{\lambda^{p-1}}\Big] = \operatorname{div}\mathbf{F} \quad \text{in } B_6 \tag{2.3}$$

with $\lambda,\ \theta > 0$. The class of these equations is the smallest one that is invariant with respect to the transformations (2.2) and that contains equations of type (2.1). Indeed, if u solves (2.3) and v is given by (2.2), then v satisfies an equation of similar form, namely, $\operatorname{div}\big[\frac{\mathbf{A}'(y,\lambda'\theta' v,\lambda'\nabla v)}{\lambda'^{p-1}}\big] = \operatorname{div}\mathbf{F}'$ in $B_{\frac{6}{r}}$ with $\mathbf{A}'(y,z,\xi) := \mathbf{A}(ry,z,\xi)$, $\mathbf{F}'(y) := \mathbf{F}(ry)/\mu^{p-1}$, $\lambda' := \mu\lambda$ and $\theta' := r\theta$.

Let us give the precise definition of weak solutions that is used throughout the paper.

Definition 2.3. *Let* $\mathbf{F} \in L^{\frac{p}{p-1}}(B_6;\mathbb{R}^n)$. *A function* $u \in W^{1,p}_{loc}(B_6)$ *is called a weak solution of* (2.3) *if* $u(x) \in \frac{1}{\lambda\theta}\overline{\mathbb{K}}$ *for a.e.* $x \in B_6$ *and*

$$\int_{B_6}\Big\langle\frac{\mathbf{A}(x,\lambda\theta u,\lambda\nabla u)}{\lambda^{p-1}},\nabla\varphi\Big\rangle\,dx = \int_{B_6}\langle\mathbf{F},\nabla\varphi\rangle\,dx \qquad \forall\varphi \in W^{1,p}_0(B_6).$$

Our main result on the interior regularity is the following theorem:

Theorem 2.4. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4), *and* $M_0 > 0$. *For any* $q > p$, *there exists a constant* $\delta = \delta(p,q,n,\Lambda,\eta,\mathbb{K},M_0) > 0$ *such that: if* $\lambda > 0$, $0 < \theta \le 1$,

$$\sup_{0<\rho\le 3}\sup_{y\in B_1} dist\Big(\mathbf{A},\mathbb{G}_{B_\rho(y)}(\eta)\Big) \le \delta,$$

and u *is a weak solution of* (2.3) *satisfying* $\|u\|_{L^\infty(B_5)} \le \frac{M_0}{\lambda\theta}$, *then*

$$\|\nabla u\|_{L^q(B_1)} \le C(p,q,n,\Lambda,\eta,\mathbb{K},M_0)\Big(\|u\|_{L^p(B_6)} + \||\mathbf{F}|^{\frac{1}{p-1}}\|_{L^q(B_6)}\Big). \tag{2.4}$$

By taking $\lambda = \theta = 1$ in Theorem 2.4, we then obtain $W^{1,q}$-estimates for weak solutions to original equation (2.1). Another observation is that any function $\mathbf{f} \in L^p(B_6)$ can be written in the form $\mathbf{f} = \operatorname{div}\nabla\psi$, where $\psi \in W^{1,2}_0(B_6)$ is the weak solution to the Dirichlet problem

$$\begin{cases} \Delta\psi &= \mathbf{f} \quad \text{in } B_6,\\ \psi &= 0 \quad \text{on } \partial B_6. \end{cases}$$

Moreover by the standard estimate using Riesz potential (see [29, page 195] for an explanation), we have when $1 < l < n$ that

$$\|\nabla\psi\|_{L^{\frac{nl}{n-l}}(B_6)} \le C(n,l)\,\|\mathbf{f}\|_{L^l(B_6)}.$$

These facts together with Theorem 2.4 yield:

Corollary 2.5. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4), *and* $M_0 > 0$. *For any* $\max\{1,\frac{np}{np+p-n}\} < l < n$, *there exists a constant* $\delta = \delta(p,l,n,\Lambda,\eta,\mathbb{K},M_0) > 0$ *such that: if* $\lambda > 0$, $0 < \theta \le 1$,

$$\sup_{0<\rho\le 3}\sup_{y\in B_1} dist\Big(\mathbf{A},\mathbb{G}_{B_\rho(y)}(\eta)\Big) \le \delta,$$

and u *is a weak solution of*

$$div\Big[\frac{\mathbf{A}(x,\lambda\theta u,\lambda\nabla u)}{\lambda^{p-1}}\Big] = div\,\mathbf{F} + \mathbf{f} \quad in \quad B_6$$

satisfying $\|u\|_{L^\infty(B_5)} \le \frac{M_0}{\lambda\theta}$, *then*

$$\|\nabla u\|^{p-1}_{L^{\frac{nl(p-1)}{n-l}}(B_1)} \le C(p,l,n,\Lambda,\eta,\mathbb{K},M_0)\Big(\|u\|^{p-1}_{L^p(B_6)} + \|\mathbf{F}\|_{L^{\frac{nl}{n-l}}(B_6)} + \|\mathbf{f}\|_{L^l(B_6)}\Big).$$

3. Some elementary estimates

In this section we derive some elementary estimates which will be used later. For the next lemma we only consider the case $1 < p < 2$ since (1.2) obviously yields a better estimate when $p \geq 2$.

Lemma 3.1. *Let $U \subset \mathbb{R}^n$ be a bounded open set. Assume that $\mathbf{A} : U \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies* (1.2) *for a.e. x in U and for some $1 < p < 2$. Then for any functions $u, v \in W^{1,p}(U)$ and any nonnegative function $\phi \in C(\overline{U})$, we have*

$$(3.1)\quad (1-\tau)\int_U |\nabla u - \nabla v|^p \phi\, dx \leq \tau \int_U |\nabla u|^p \phi\, dx + \Lambda 2^{2-p}\frac{p}{2}\Big(\frac{2-p}{2^{2-p}\tau}\Big)^{\frac{2-p}{p}} \int_U \langle \mathbf{A}(x,u,\nabla u) - \mathbf{A}(x,u,\nabla v), \nabla u - \nabla v\rangle \phi\, dx$$

for every $\tau > 0$.

Proof. Since $|\xi| + |\eta| \leq 2(|\xi| + |\xi - \eta|)$ and $1 < p < 2$, we have from (1.2) that

$$(3.2)\qquad \langle \mathbf{A}(x,z,\xi) - \mathbf{A}(x,z,\eta), \xi - \eta\rangle \geq \Lambda^{-1}2^{p-2}(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^n.$$

Using Young's inequality, the assumption $1 < p < 2$ and (3.2), we then obtain

$$\begin{aligned}
&\int_U |\nabla u - \nabla v|^p \phi\, dx = \int_U \big[(|\nabla u| + |\nabla u - \nabla v|)\phi^{\frac{1}{p}}\big]^{\frac{p(2-p)}{2}}\big[(|\nabla u| + |\nabla u - \nabla v|)^{\frac{p(p-2)}{2}}|\nabla u - \nabla v|^p \phi^{\frac{p}{2}}\big]\, dx\\
&\leq \frac{\tau}{2^{p-1}}\int_U (|\nabla u| + |\nabla u - \nabla v|)^p \phi\, dx + \frac{p}{2}\Big(\frac{2-p}{2^{2-p}\tau}\Big)^{\frac{2-p}{p}}\int_U (|\nabla u| + |\nabla u - \nabla v|)^{p-2}|\nabla u - \nabla v|^2 \phi\, dx\\
&\leq \tau \int_U |\nabla u|^p \phi\, dx + \tau \int_U |\nabla u - \nabla v|^p \phi\, dx\\
&\quad + \Lambda 2^{2-p}\frac{p}{2}\Big(\frac{2-p}{2^{2-p}\tau}\Big)^{\frac{2-p}{p}}\int_U \langle \mathbf{A}(x,u,\nabla u) - \mathbf{A}(x,u,\nabla v), \nabla u - \nabla v\rangle \phi\, dx.
\end{aligned}$$

This gives the lemma as desired. □

The next two results are about basic L^p-estimates for gradients of weak solutions.

Proposition 3.2. *Assume that $\mathbf{A} : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies* (1.2) *and* (1.3). *Let $w \in W^{1,p}(B_3)$ be a weak solution of*

$$\begin{cases} \mathrm{div}\, \mathbf{A}(x,w,\nabla w) = \mathrm{div}\, \mathbf{F} & \text{in } B_3,\\ w = \varphi & \text{on } \partial B_3, \end{cases}$$

where $\varphi \in W^{1,p}(B_3)$. Then

$$\int_{B_3} |\nabla w|^p\, dx \leq C(p,n,\Lambda)\Big(\int_{B_3} |\nabla \varphi|^p\, dx + \int_{B_3} |\mathbf{F}|^{\frac{p}{p-1}}\, dx\Big).$$

Proof. By using $w - \varphi$ as a test function, we get

$$\int_{B_3} \langle \mathbf{A}(x,w,\nabla w), \nabla w - \nabla \varphi\rangle\, dx = \int_{B_3} \langle \mathbf{F}, \nabla w - \nabla \varphi\rangle\, dx$$

which can be rewritten as

$$\int_{B_3} \langle \mathbf{A}(x,w,\nabla w) - \mathbf{A}(x,w,0), \nabla w\rangle dx = \int_{B_3} \langle \mathbf{A}(x,w,\nabla w), \nabla \varphi\rangle dx + \int_{B_3} \langle \mathbf{F}, \nabla w\rangle dx - \int_{B_3} \langle \mathbf{F}, \nabla \varphi\rangle dx.$$

On the other hand, it follows from (1.2) that

$$\Lambda^{-1}\int_{B_3} |\nabla w|^p\, dx \leq \int_{B_3} \langle \mathbf{A}(x,w,\nabla w) - \mathbf{A}(x,w,0), \nabla w\rangle\, dx.$$

Therefore, we obtain

$$\Lambda^{-1}\int_{B_3}|\nabla w|^p\,dx \le \Lambda\int_{B_3}|\nabla w|^{p-1}|\nabla\varphi|\,dx + \int_{B_3}|\mathbf{F}||\nabla w|\,dx + \int_{B_3}|\mathbf{F}||\nabla\varphi|\,dx.$$

From this and by applying Young's inequality, we deduce the conclusion of the lemma. □

Lemma 3.3. *Assume that* $\mathbf{A}: B_4\times\overline{\mathbb{K}}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *satisfies* (1.2) *and* (1.3). *Let* $u \in W^{1,p}_{loc}(B_4)$ *be a weak solution of*

$$div\,\mathbf{A}(x,u,\nabla u) = div\,\mathbf{F} \quad in \quad B_4. \tag{3.3}$$

Then

$$\int_{B_3}|\nabla u|^p\,dx \le C(p,n,\Lambda)\Big(\int_{B_4}|u|^p\,dx + \int_{B_4}|\mathbf{F}|^{\frac{p}{p-1}}\,dx\Big).$$

Proof. Let $\varphi\in C_0^\infty(B_4)$ be the standard nonnegative cut-off function which is 1 on B_3. Then, by multiplying the equation by $\varphi^p u$ and using integration by parts we get

$$\begin{aligned}\int_{B_4}\langle\mathbf{A}(x,u,\nabla u)-\mathbf{A}(x,u,0),\nabla u\rangle\varphi^p\,dx &= -p\int_{B_4}\langle\mathbf{A}(x,u,\nabla u),\nabla\varphi\rangle\varphi^{p-1}u\,dx\\ &\quad+\int_{B_4}\langle\mathbf{F},\nabla u\rangle\varphi^p\,dx + p\int_{B_4}\langle\mathbf{F},\nabla\varphi\rangle\varphi^{p-1}u\,dx.\end{aligned}$$

Therefore, it follows from (1.2) and (1.3) that

$$\Lambda^{-1}\int_{B_4}|\nabla u|^p\varphi^p\,dx \le p\Lambda\int_{B_4}|\nabla u|^{p-1}|\nabla\varphi|\varphi^{p-1}|u|\,dx + \int_{B_4}|\mathbf{F}||\nabla u|\varphi^p\,dx + p\int_{B_4}|\mathbf{F}||\nabla\varphi|\varphi^{p-1}|u|\,dx.$$

This together with Young's inequality yields the lemma. □

We end the section with a result giving a bound on the L^p-norm of the difference between two gradients of weak solutions.

Lemma 3.4. *Assume* $\mathbf{A}: B_4\times\overline{\mathbb{K}}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and* $\mathbf{a}: B_3\times\overline{\mathbb{K}}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *satisfy* (1.2) *and* (1.3). *Let* $u\in W^{1,p}_{loc}(B_4)$ *be a weak solution of* (3.3) *and* $v\in W^{1,p}(B_3)$ *be a weak solution of*

$$\begin{cases} div\,\mathbf{a}(x,v,\nabla v) &= 0 \quad in \quad B_3,\\ v &= u \quad on \quad \partial B_3.\end{cases}$$

Then

$$\int_{B_3}|\nabla u-\nabla v|^p\,dx \le C(p,n,\Lambda)\int_{B_4}(|u|^p + |\mathbf{F}|^{\frac{p}{p-1}})\,dx. \tag{3.4}$$

Moreover,

$$\int_{B_3}|v|^p\,dx \le C(p,n,\Lambda)\int_{B_4}(|u|^p + |\mathbf{F}|^{\frac{p}{p-1}})\,dx. \tag{3.5}$$

Proof. By using $u-v$ as a test function in the equations for u and v, we get

$$-\int_{B_3}\langle\mathbf{a}(x,v,\nabla v),\nabla u-\nabla v\rangle dx = -\int_{B_3}\langle\mathbf{A}(x,u,\nabla u),\nabla u-\nabla v\rangle\,dx + \int_{B_3}\langle\mathbf{F},\nabla u-\nabla v\rangle dx.$$

This gives

$$\begin{aligned}J &:= \int_{B_3}\langle\mathbf{a}(x,v,\nabla u)-\mathbf{a}(x,v,\nabla v),\nabla u-\nabla v\rangle\,dx\\ &= \int_{B_3}\langle\mathbf{a}(x,v,\nabla u)-\mathbf{A}(x,u,\nabla u),\nabla u-\nabla v\rangle\,dx + \int_{B_3}\langle\mathbf{F},\nabla u-\nabla v\rangle\,dx.\end{aligned}$$

It follows from this and (1.3) that

$$J \leq 2\Lambda \int_{B_3} |\nabla u|^{p-1} |\nabla u - \nabla v| \, dx + \int_{B_3} |\mathbf{F}||\nabla u - \nabla v| \, dx.$$

Moreover, Lemma 3.1 and (1.2) imply that

$$c \int_{B_3} |\nabla u - \nabla v|^p \, dx - c^{-1} \int_{B_3} |\nabla u|^p \, dx \leq J.$$

Therefore, we conclude that

$$c \int_{B_3} |\nabla u - \nabla v|^p \, dx \leq c^{-1} \int_{B_3} |\nabla u|^p \, dx + 2\Lambda \int_{B_3} |\nabla u|^{p-1} |\nabla u - \nabla v| \, dx + \int_{B_3} |\mathbf{F}||\nabla u - \nabla v| \, dx.$$

We infer from this and Young's inequality that

$$\int_{B_3} |\nabla u - \nabla v|^p \, dx \leq C(p, n, \Lambda) \int_{B_3} \left(|\nabla u|^p + |\mathbf{F}|^{\frac{p}{p-1}} \right) dx.$$

This together with Lemma 3.3 yields (3.4). On the other hand, (3.5) is a consequence of (3.4) and the estimate

$$\int_{B_3} |v|^p \, dx \leq 2^{p-1} \Big[\int_{B_3} |u|^p \, dx + \int_{B_3} |u - v|^p \, dx \Big] \leq C(p, n) \Big[\int_{B_3} |u|^p \, dx + \int_{B_3} |\nabla u - \nabla v|^p \, dx \Big].$$

□

4. Approximating solutions

The goal of this section is to prove a result allowing us to compare solutions originating from two different equations.

4.1. **Strong compactness of the class $\mathbb{G}$ of vector fields.** In this subsection we give some elementary arguments showing that the class of vector fields $\mathbb{G}_{B_3}(\eta)$ is relatively compact with respect to the pointwise convergence. Let us first recall the sequential Bocce criterion in [1].

Definition 4.1. *We say that a sequence $\{f_k\}$ in $L^1(B_3; \mathbb{R}^n)$ satisfies the sequential Bocce criterion if for each subsequence $\{f_{k_j}\}$ of $\{f_k\}$, each $\epsilon > 0$ and each measurable set $E \subset B_3$ with $|E| > 0$, there exists a measurable set $A \subset E$ with $|A| > 0$ such that*

$$\liminf_{j \to \infty} \fint_A |f_{k_j}(x) - (f_{k_j})_A| \, dx < \epsilon. \tag{4.1}$$

The following result is a special case of [1, Theorem 2.3].

Theorem 4.2. *(Theorem 2.3 in [1]) Let $\{f_k\}$ be a sequence in $L^1(B_3; \mathbb{R}^n)$. Then $\{f_k\}$ converges strongly to f in $L^1(B_3; \mathbb{R}^n)$ if and only if*

(1) *$\{f_k\}$ converges weakly to f in $L^1(B_3; \mathbb{R}^n)$.*
(2) *$\{f_k\}$ satisfies the sequential Bocce criterion.*

An application of Theorem 4.2 gives:

Lemma 4.3. *Let $\eta : [0, \infty) \to \mathbb{R}$ be a function satisfying $\lim_{r \to 0^+} \eta(r) = 0$. Suppose $\{f_k\}$ converges weakly to f in $L^1(B_3; \mathbb{R}^n)$, and*

$$\sup_k \sup_{x_0 :\, B_r(x_0) \subset B_3} \fint_{B_r(x_0)} |f_k(x) - (f_k)_{B_r(x_0)}| \, dx \leq \eta(r)$$

for all $r > 0$ sufficiently small. Then $\{f_k\}$ converges strongly to f in $L^1(B_3; \mathbb{R}^n)$.

Proof. By Theorem 4.2, it is enough to check that $\{f_k\}$ satisfies the sequential Bocce criterion. For this, let $\epsilon > 0$ and let $E \subset B_3$ be a measurable set with $|E| > 0$. Then by the Lebesgue differentiation theorem, there exists $x_0 \in E$ such that

$$\lim_{r\to 0^+} \frac{|E \cap B_r(x_0)|}{|B_r(x_0)|} = 1 \quad \text{and} \quad \lim_{r\to 0^+} \fint_{B_r(x_0)} f = \lim_{r\to 0^+} \fint_{B_r(x_0)} f\chi_E = f(x_0). \tag{4.2}$$

For all $r > 0$ small, we have with $A_r := E \cap B_r(x_0)$ that

$$\begin{aligned} \fint_{A_r} |f_k(x) - (f_k)_{A_r}|\, dx &\le \fint_{A_r} |f_k(x) - (f_k)_{B_r(x_0)}|\, dx + \Big| \fint_{B_r(x_0)} f_k - \fint_{A_r} f_k \Big| \\ &\le \frac{|B_r(x_0)|}{|A_r|} \eta(r) + \Big| \fint_{B_r(x_0)} f_k - \fint_{A_r} f_k \Big| \quad \forall k. \end{aligned}$$

It follows by taking $k \to \infty$ and using the weak convergence of $\{f_k\}$ to f that

$$\begin{aligned} \limsup_{k\to\infty} \fint_{A_r} |f_k(x) - (f_k)_{A_r}|\, dx &\le \frac{|B_r(x_0)|}{|A_r|} \eta(r) + \Big| \fint_{B_r(x_0)} f - \fint_{A_r} f \Big| \\ &= \frac{|B_r(x_0)|}{|A_r|} \eta(r) + \Big| \fint_{B_r(x_0)} f - \frac{|B_r(x_0)|}{|A_r|} \fint_{B_r(x_0)} f\chi_E \Big| \quad \forall r > 0 \text{ small}. \end{aligned}$$

Thanks to (4.2) and the assumption $\lim_{r\to 0^+} \eta(r) = 0$, we can choose $r > 0$ sufficiently small such that the above right hand side is less than ϵ. Thus $\{f_k\}$ satisfies the sequential Bocce criterion and the proof is complete. ◻

The strong compactness of $\mathbb{G}_{B_3}(\eta)$ is given by the next result. This technical lemma will be used in Subsection 4.2.

Lemma 4.4. *For each positive integer k, let $\mathbf{a}_k : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector field satisfying conditions* (1.3)–(1.4) *and (**H1**)–(**H2**). Then there exist a subsequence still denoted by $\{\mathbf{a}_k\}$ and a vector field $\mathbf{a} : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mathbf{a}_k(x, z, \xi) \to \mathbf{a}(x, z, \xi) \quad \text{for a.e. } x \in B_3 \text{ and for all } (z, \xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n.$$

Moreover, $\mathbf{a}$ is continuous in the ξ variable.

Proof. We first observe the following.

Claim: For any sequence $\{\xi_n\} \subset \mathbb{R}^n$ with $\xi_n \to \xi$, there exists a constant $C > 0$ depending only on ξ, p and Λ such that

$$\sup_k \sup_{(x,z)\in B_3\times\overline{\mathbb{K}}} |\mathbf{a}_k(x, z, \xi_m) - \mathbf{a}_k(x, z, \xi_n)| \le C \max\{|\xi_m - \xi_n|, |\xi_m - \xi|^{p-1} + |\xi_n - \xi|^{p-1}\} \tag{4.3}$$

for all m and n sufficiently large. Since the case $\xi = 0$ is obvious from (1.3), we only need to prove the claim for the case $\xi \neq 0$. Then there exists $N_0 \in \mathbb{N}$ such that $\xi_k \in B(\xi, \frac{|\xi|}{2})$ for all $k \ge N_0$. Hence we get from the mean value property and (**H1**) that

$$\begin{aligned} |\mathbf{a}_k(x, z, \xi_m) - \mathbf{a}_k(x, z, \xi_n)| &= |\partial_\xi \mathbf{a}_k(x, z, \alpha\xi_m + (1-\alpha)\xi_n)|\, |\xi_m - \xi_n| \\ &\le \Lambda |\alpha\xi_m + (1-\alpha)\xi_n|^{p-2}\, |\xi_m - \xi_n| \le C|\xi_m - \xi_n| \quad \forall n, m \ge N_0, \end{aligned}$$

giving the claim.

Next let $(z, \xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n$ be fixed and define

$$\hat{\mathbf{a}}_k(x) := \mathbf{a}_k(x, z, \xi) \quad \text{for} \quad x \in B_3.$$

Then $\{\hat{\mathbf{a}}_k\}$ is bounded in $L^\infty(B_3)$ by (1.3) and so there exists a subsequence depending on (z, ξ) and $\hat{\mathbf{a}} \in L^\infty(B_3)$ such that

$$\hat{\mathbf{a}}_k \rightharpoonup \hat{\mathbf{a}} \quad \text{weakly-* in } L^\infty(B_3; \mathbb{R}^n).$$

Hence it follows from condition (**H2**) and Lemma 4.3 that $\hat{\mathbf{a}}_k \longrightarrow \hat{\mathbf{a}}$ strongly in $L^1(B_3;\mathbb{R}^n)$. Thus we can extract a further subsequence, still denoted by $\{\hat{\mathbf{a}}_k\}$, such that $\hat{\mathbf{a}}_k(x) \to \hat{\mathbf{a}}(x)$ for a.e. $x \in B_3$.

Therefore, we infer from the diagonal process that there exist a subsequence $\{\mathbf{a}_k\}$ and a vector field $\mathbf{a} : B_3 \times (\overline{\mathbb{K}} \cap \mathbb{Q}) \times (\mathbb{R}^n \cap \mathbb{Q}^n) \to \mathbb{R}^n$ satisfying

$$\mathbf{a}_k(x,z,\xi) \to \mathbf{a}(x,z,\xi)$$

for a.e. $x \in B_3$ and for all $(z,\xi) \in (\overline{\mathbb{K}} \cap \mathbb{Q}) \times (\mathbb{R}^n \cap \mathbb{Q}^n)$. We are going to show that $\mathbf{a}$ admits an extension on $B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n$ with the property

$$\mathbf{a}_k(x,z,\xi) \to \mathbf{a}(x,z,\xi) \quad \text{for a.e. } x \in B_3 \text{ and for all } (z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n. \tag{4.4}$$

Let $(x,z,\xi) \in B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n$ and take a sequence $\{(z_n,\xi_n)\} \subset (\overline{\mathbb{K}} \cap \mathbb{Q}) \times (\mathbb{R}^n \cap \mathbb{Q}^n)$ such that $(z_n,\xi_n) \to (z,\xi)$. By using (1.4) and (4.3) we obtain for all m,n large that

$$\begin{aligned}|\mathbf{a}_k(x,z_m,\xi_m) - \mathbf{a}_k(x,z_n,\xi_n)| &\le |\mathbf{a}_k(x,z_m,\xi_m) - \mathbf{a}_k(x,z_n,\xi_m)| + |\mathbf{a}_k(x,z_n,\xi_m) - \mathbf{a}_k(x,z_n,\xi_n)| \\ &\le \Lambda |z_m - z_n|\,|\xi_m|^{p-1} + C \max\{|\xi_m - \xi_n|,\ |\xi_m - \xi|^{p-1} + |\xi_n - \xi|^{p-1}\} \quad \forall k.\end{aligned}$$

It follows by taking $k \to \infty$ that

$$|\mathbf{a}(x,z_m,\xi_m) - \mathbf{a}(x,z_n,\xi_n)| \le \Lambda |z_m - z_n|\,|\xi_m|^{p-1} + C \max\{|\xi_m - \xi_n|,\ |\xi_m - \xi|^{p-1} + |\xi_n - \xi|^{p-1}\} \tag{4.5}$$

for all m,n sufficiently large. Thus, $\{\mathbf{a}(x,z_n,\xi_n)\}$ is a Cauchy sequence in $\mathbb{R}^n$ and we define

$$\mathbf{a}(x,z,\xi) := \lim_{n\to\infty} \mathbf{a}(x,z_n,\xi_n).$$

We note that this definition of $\mathbf{a}(x,z,\xi)$ is independent of the choice of the sequence $\{(z_n,\xi_n)\}$. Indeed, if $\{(z_n',\xi_n')\}$ is another sequence in $(\overline{\mathbb{K}} \cap \mathbb{Q}) \times (\mathbb{R}^n \cap \mathbb{Q}^n)$ satisfying $(z_n',\xi_n') \to (z,\xi)$, then by the same arguments leading to (4.5) we have

$$|\mathbf{a}(x,z_n,\xi_n) - \mathbf{a}(x,z_n',\xi_n')| \le \Lambda |z_n - z_n'|\,|\xi_n|^{p-1} + C \max\{|\xi_n - \xi_n'|,\ |\xi_n - \xi|^{p-1} + |\xi_n' - \xi|^{p-1}\}.$$

Therefore, the convergent sequences $\{\mathbf{a}(x,z_n,\xi_n)\}$ and $\{\mathbf{a}(x,z_n',\xi_n')\}$ have the same limit.

Let us now verify (4.4). Let $(x,z,\xi) \in B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n$ be arbitrary. Take $\{(z_n,\xi_n)\} \subset (\overline{\mathbb{K}} \cap \mathbb{Q}) \times (\mathbb{R}^n \cap \mathbb{Q}^n)$ be such that $(z_n,\xi_n) \to (z,\xi)$. Then the triangle inequality gives

$$\begin{aligned}|\mathbf{a}_k(x,z,\xi) - \mathbf{a}(x,z,\xi)| &\le |\mathbf{a}_k(x,z,\xi) - \mathbf{a}_k(x,z,\xi_n)| + |\mathbf{a}_k(x,z,\xi_n) - \mathbf{a}_k(x,z_n,\xi_n)| \\ &\quad + |\mathbf{a}_k(x,z_n,\xi_n) - \mathbf{a}(x,z_n,\xi_n)| + |\mathbf{a}(x,z_n,\xi_n) - \mathbf{a}(x,z,\xi)| \quad \forall n.\end{aligned}$$

Moreover, it follows from (4.3) by letting $m \to \infty$ that

$$|\mathbf{a}_k(x,z,\xi) - \mathbf{a}_k(x,z,\xi_n)| \le C \max\{|\xi - \xi_n|,\ |\xi_n - \xi|^{p-1}\}. \tag{4.6}$$

Thus, we deduce that

$$\begin{aligned}|\mathbf{a}_k(x,z,\xi) - \mathbf{a}(x,z,\xi)| &\le C \max\{|\xi - \xi_n|,\ |\xi_n - \xi|^{p-1}\} + \Lambda |\xi_n|^{p-1} |z - z_n| \\ &\quad + |\mathbf{a}_k(x,z_n,\xi_n) - \mathbf{a}(x,z_n,\xi_n)| + |\mathbf{a}(x,z_n,\xi_n) - \mathbf{a}(x,z,\xi)| \quad \forall n \ge N_0,\end{aligned}$$

where N_0 depends on ξ but independent of k. Consequently,

$$\begin{aligned}\limsup_{k\to\infty} |\mathbf{a}_k(x,z,\xi) - \mathbf{a}(x,z,\xi)| &\le C \max\{|\xi - \xi_n|,\ |\xi_n - \xi|^{p-1}\} + \Lambda |\xi_n|^{p-1} |z - z_n| \\ &\quad + |\mathbf{a}(x,z_n,\xi_n) - \mathbf{a}(x,z,\xi)|\end{aligned}$$

for all $n \ge N_0$. Letting $n \to \infty$, we conclude that $\mathbf{a}_k(x,z,\xi) \to \mathbf{a}(x,z,\xi)$ and hence (4.4) holds true.

It remains to show that $\mathbf{a}$ is continuous in the ξ variable. To see this, let $\xi_n \to \xi$ in $\mathbb{R}^n$. Then (4.6) is satisfied for all k and so by letting k tend to infinity and using (4.4) we obtain

$$|\mathbf{a}(x,z,\xi) - \mathbf{a}(x,z,\xi_n)| \le C \max\{|\xi - \xi_n|,\ |\xi_n - \xi|^{p-1}\} \qquad \text{for all large } n.$$

Therefore for a.e. $x \in B_3$ and all $z \in \overline{\mathbb{K}}$, the vector field $\xi \mapsto \mathbf{a}(x,z,\xi)$ is continuous on $\mathbb{R}^n$. □

4.2. **An approximation lemma.** We begin this subsection with a result needed for the proof of the approximation lemma (Lemma 4.6).

Lemma 4.5. *Let $\omega : [0,\infty) \to \mathbb{R}$ be a function satisfying $\lim_{s\to 0^+} \omega(s) = \omega(0) = 0$. For each k, let $\mathbf{A}_k : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \to \mathbb{R}^n$ be such that for a.e. $x \in B_3$ there hold*

$$\langle \mathbf{A}_k(x,z,\xi) - \mathbf{A}_k(x,z,\eta), \xi - \eta\rangle \geq 0 \qquad \forall z \in \overline{\mathbb{K}} \text{ and } \forall \xi,\eta \in \mathbb{R}^n, \tag{4.7}$$

$$|\mathbf{A}_k(x,z,\xi)| \leq \Lambda(1+|\xi|^2)^{\frac{p-1}{2}} \qquad \forall (z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n, \tag{4.8}$$

$$|\mathbf{A}_k(x,z_1,\xi) - \mathbf{A}_k(x,z_2,\xi)| \leq \omega(|z_1-z_2|)(1+|\xi|^2)^{\frac{p-1}{2}} \quad \forall z_1,z_2 \in \overline{\mathbb{K}} \text{ and } \forall \xi \in \mathbb{R}^n. \tag{4.9}$$

Suppose in addition that $\mathbf{A}_k(x,z,\xi) \to \mathbf{A}(x,z,\xi)$ for a.e. $x \in B_3$ and for all $(z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n$, where $\mathbf{A} : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in the ξ variable. Let $u^k \in W^{1,p}(B_3)$ be a weak solution to

$$\operatorname{div} \mathbf{A}_k(x,m^k,\nabla u^k) = \operatorname{div} \mathbf{F}_k \quad \text{in} \quad B_3 \tag{4.10}$$

with $m^k \in L^1(B_3)$ satisfying $m^k(x) \in \overline{\mathbb{K}}$ for a.e. $x \in B_3$. Assume that $u^k \to u$ strongly in $L^p(B_3)$, $\nabla u^k \rightharpoonup \nabla u$ weakly in $L^p(B_3)$, $m^k \to m$ a.e. in B_3, $\mathbf{F}_k \to 0$ strongly in $L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n)$, and

$$\mathbf{A}_k(x,m^k,\nabla u^k) \rightharpoonup \zeta \quad \text{weakly in } L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n) \quad \text{for some} \quad \zeta \in L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n). \tag{4.11}$$

Then we have

$$\zeta(x) = \mathbf{A}(x,m(x),\nabla u(x)) \quad \text{for a.e. } x \in B_3.$$

Proof. We shall use Minty–Browder's technique which employs monotonicity to justify passing to weak limits within a nonlinearity (see [15, 23, 26]). This technique was also used in [3]. Let $\phi \in C_0^\infty(B_3)$ be a nonnegative function. Then for any function $v \in W^{1,p}(B_3)$, we have from (4.7) that

$$\int_{B_3} \Big\langle \mathbf{A}_k(x,m^k,\nabla u^k) - \mathbf{A}_k(x,m^k,\nabla v), \nabla u^k - \nabla v\Big\rangle \phi\, dx \geq 0$$

which can be rewritten as

$$\begin{aligned}\int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla u^k), \nabla u^k\rangle \phi\, dx - \int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla u^k), \nabla v\rangle \phi\, dx \\ - \int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla v), \nabla u^k - \nabla v\rangle \phi\, dx \geq 0.\end{aligned} \tag{4.12}$$

By using $u^k\phi$ as a test function for (4.10), we see that the first term in (4.12) is the same as

$$-\int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla u^k), \nabla \phi\rangle u^k dx + \int_{B_3} \langle \mathbf{F}_k, \phi\nabla u^k + u^k \nabla\phi\rangle\, dx.$$

Therefore, inequality (4.12) becomes

$$\begin{aligned}&-\int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla u^k), \nabla \phi\rangle u^k dx + \int_{B_3} \langle \mathbf{F}_k, \phi\nabla u^k + u^k \nabla\phi\rangle\, dx \\ &-\int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla u^k), \nabla v\rangle \phi dx - \int_{B_3} \langle \mathbf{A}_k(x,m^k,\nabla v), \nabla u^k - \nabla v\rangle \phi\, dx \geq 0.\end{aligned} \tag{4.13}$$

Notice that since

$$\begin{aligned}|\mathbf{A}_k(x,m^k,\nabla v) - \mathbf{A}(x,m,\nabla v)| &\leq |\mathbf{A}_k(x,m^k,\nabla v) - \mathbf{A}_k(x,m,\nabla v)| + |\mathbf{A}_k(x,m,\nabla v) - \mathbf{A}(x,m,\nabla v)| \\ &\leq \omega(|m^k-m|)(1+|\nabla v|^2)^{\frac{p-1}{2}} + |\mathbf{A}_k(x,m,\nabla v) - \mathbf{A}(x,m,\nabla v)|,\end{aligned}$$

we get $\mathbf{A}_k(x, m^k, \nabla v) \to \mathbf{A}(x, m, \nabla v)$ for a.e. $x \in B_3$. Hence we conclude from condition (4.8) and the Lebesgue's dominated convergence theorem that $\mathbf{A}_k(x, m^k, \nabla v) \to \mathbf{A}(x, m, \nabla v)$ strongly in $L^{\frac{p}{p-1}}(B_3; \mathbb{R}^n)$. Therefore,

$$\lim_{k\to\infty} \int_{B_3} \langle \mathbf{A}_k(x, m^k, \nabla v), \nabla u^k - \nabla v\rangle \phi \, dx = \int_{B_3} \langle \mathbf{A}(x, m, \nabla v), \nabla u - \nabla v\rangle \phi \, dx.$$

Using this and assumption (4.11), we can pass to the limits in (4.13) to obtain

$$-\int_{B_3} \langle \zeta, \nabla\phi\rangle u \, dx - \int_{B_3} \langle \zeta, \nabla v\rangle \phi \, dx - \int_{B_3} \langle \mathbf{A}(x, m, \nabla v), \nabla u - \nabla v\rangle \phi \, dx \geq 0. \tag{4.14}$$

On the other hand, by choosing $u\phi$ as a test function for equation (4.10) and passing to the limits, we get $\int_{B_3} \langle \zeta, \nabla(u\phi)\rangle \, dx = 0$ which yields $-\int_{B_3} \langle \zeta, \nabla\phi\rangle u \, dx = \int_{B_3} \langle \zeta, \nabla u\rangle \phi \, dx$. Hence we can rewrite (4.14) as

$$\int_{B_3} \langle \zeta - \mathbf{A}(x, m, \nabla v), \nabla u - \nabla v\rangle \phi \, dx \geq 0.$$

By taking $v = u \pm \alpha w$ and letting $\alpha \to 0^+$, one easily deduces from the above inequality and the assumption $\mathbf{A}$ being continuous in the ξ variable that

$$\int_{B_3} \langle \zeta - \mathbf{A}(x, m, \nabla u), \nabla w\rangle \phi \, dx = 0$$

for all functions $w \in W^{1,p}(B_3)$ and all nonnegative functions $\phi \in C_0^\infty(B_3)$. It then follows that $\zeta = \mathbf{A}(x, m, \nabla u)$ a.e. in B_3. □

The following approximation lemma plays a central role in our proof of Theorem 2.4. It is crucial that the constant $\delta > 0$ is independent of the parameters λ and θ. We shall prove it by extending the compactness argument used in [17, Lemma 2.11] and for this purpose we define

$$d_{\mathbf{A},\mathbf{a}}(x) := \sup_{z\in\mathbb{K}} \sup_{\xi\neq 0} \frac{|\mathbf{A}(x, z, \xi) - \mathbf{a}(x, z, \xi)|}{|\xi|^{p-1}}.$$

Lemma 4.6. *Let* $\mathbf{A}$ *satisfy* (1.2)–(1.4), $\mathbf{a} \in \mathbb{G}_{B_3}(\eta)$, *and* $M_0 > 0$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *depending only on* ε, Λ, p, η, n, $\mathbb{K}$ *and* M_0 *such that: if* $\lambda > 0$, $0 < \theta \leq 1$,

$$\fint_{B_3} d_{\mathbf{A},\mathbf{a}}(x) \, dx \leq \delta, \qquad \fint_{B_4} |\mathbf{F}|^{\frac{p}{p-1}} \, dx \leq \delta,$$

and $u \in W^{1,p}_{loc}(B_4)$ *is a weak solution of*

$$div\Big[\frac{\mathbf{A}(x, \lambda\theta u, \lambda\nabla u)}{\lambda^{p-1}}\Big] = div\, \mathbf{F} \quad in \quad B_4 \tag{4.15}$$

satisfying

$$\Big(\fint_{B_4} |u|^p \, dx\Big)^{\frac{1}{p}} \leq \frac{M_0}{\lambda\theta} \quad and \quad \fint_{B_4} |\nabla u|^p \, dx \leq 1, \tag{4.16}$$

and $v \in W^{1,p}(B_3)$ *is a weak solution of*

$$\begin{cases} div\Big[\frac{\mathbf{a}(x,\lambda\theta v,\lambda\nabla v)}{\lambda^{p-1}}\Big] & = & 0 & in & B_3, \\ v & = & u & on & \partial B_3, \end{cases} \tag{4.17}$$

then

$$\int_{B_3} |u - v|^p \, dx \leq \varepsilon^p. \tag{4.18}$$

Proof. We prove (4.18) by contradiction. Suppose that estimate (4.18) is not true. Then there exist ε_0, p, Λ, η, n, $\mathbb{K}$, M_0, sequences of positive numbers $\{\lambda_k\}_{k=1}^\infty$ and $\{\theta_k\}_{k=1}^\infty$ with $0<\theta_k\le 1$, sequences $\{\mathbf{A}_k\}_{k=1}^\infty$ and $\{\mathbf{a}_k\}_{k=1}^\infty$ with $\mathbf{A}_k$ satisfying the structural conditions (1.2)–(1.4) and $\mathbf{a}_k\in\mathbb{G}_{B_3}(\eta)$, and sequences of functions $\{\mathbf{F}_k\}_{k=1}^\infty$, $\{u^k\}_{k=1}^\infty$ such that

$$\sup_{\xi\neq 0}\sup_{x_0:\,B_r(x_0)\subset B_3}\fint_{B_r(x_0)}\frac{\big|\mathbf{a}_k(x,z,\xi)-(\mathbf{a}_k)_{B_r(x_0)}(z,\xi)\big|}{|\xi|^{p-1}}\,dx\le\eta(z,r),$$

$$\fint_{B_3} d_{\mathbf{A}_k,\mathbf{a}_k}(x)\,dx\le\frac{1}{k},\qquad \fint_{B_4}|\mathbf{F}_k|^{\frac{p}{p-1}}\,dx\le\frac{1}{k},\tag{4.19}$$

$u^k\in W^{1,p}_{loc}(B_4)$ is a weak solution of

$$\operatorname{div}\Big[\frac{\mathbf{A}_k(x,\lambda_k\theta_k u^k,\lambda_k\nabla u^k)}{\lambda_k^{p-1}}\Big]=\operatorname{div}\mathbf{F}_k\quad\text{in}\quad B_4$$

with

$$\Big(\fint_{B_4}|u^k|^p\,dx\Big)^{\frac{1}{p}}\le\frac{M_0}{\lambda_k\theta_k}\quad\text{and}\quad\fint_{B_4}|\nabla u^k|^p\,dx\le 1,\tag{4.20}$$

$$\int_{B_3}|u^k-v^k|^p\,dx>\varepsilon_0^p\quad\text{for all }k.\tag{4.21}$$

Here $v^k\in W^{1,p}(B_3)$ is a weak solution of

$$\begin{cases}\operatorname{div}\Big[\frac{\mathbf{a}_k(x,\lambda_k\theta_k v^k,\lambda_k\nabla v^k)}{\lambda_k^{p-1}}\Big] &= 0 \quad\text{in}\quad B_3,\\ v^k &= u^k\quad\text{on}\quad\partial B_3.\end{cases}$$

Let us set

$$\alpha_k:=\lambda_k\theta_k,\quad \hat{\mathbf{A}}_k(x,z,\xi):=\frac{\mathbf{A}_k(x,z,\lambda_k\xi)}{\lambda_k^{p-1}}\quad\text{and}\quad\hat{\mathbf{a}}_k(x,z,\xi):=\frac{\mathbf{a}_k(x,z,\lambda_k\xi)}{\lambda_k^{p-1}}.$$

Then, we still have

$$d_{\hat{\mathbf{A}}_k,\hat{\mathbf{a}}_k}(x)=d_{\mathbf{A}_k,\mathbf{a}_k}(x)$$

and

$$\sup_{\xi\neq 0}\sup_{x_0:\,B_r(x_0)\subset B_3}\fint_{B_r(x_0)}\frac{\big|\hat{\mathbf{a}}_k(x,z,\xi)-(\hat{\mathbf{a}}_k)_{B_r(x_0)}(z,\xi)\big|}{|\xi|^{p-1}}\,dx\le\eta(z,r).\tag{4.22}$$

Moreover, u^k is a weak solution of

$$\operatorname{div}\hat{\mathbf{A}}_k(x,\alpha_k u^k,\nabla u^k)=\operatorname{div}\mathbf{F}_k\quad\text{in}\quad B_4,\tag{4.23}$$

and v^k is a weak solution of

$$\begin{cases}\operatorname{div}\hat{\mathbf{a}}_k(x,\alpha_k v^k,\nabla v^k) &= 0\quad\text{in}\quad B_3,\\ v^k &= u^k\quad\text{on}\quad\partial B_3.\end{cases}$$

Using Sobolev's inequality, Lemma 3.4, $\Big(\fint_{B_4}|u^k|^p\,dx\Big)^{\frac{1}{p}}\le\frac{M_0}{\alpha_k}$ and (4.19), we obtain

$$\int_{B_3}|u^k-v^k|^p\,dx\le C\int_{B_3}|\nabla u^k-\nabla v^k|^p\,dx\le C(p,n,\Lambda)\int_{B_4}\big(|u^k|^p+|\mathbf{F}_k|^{\frac{p}{p-1}}\big)\,dx\le C\Big[\big(\frac{M_0}{\alpha_k}\big)^p+k^{-1}\Big].$$

Thus we infer from (4.21) that

$$\alpha_k\le\frac{M_0}{C^{-1}\varepsilon_0-k^{\frac{-1}{p}}},$$

and so the sequence $\{\alpha_k\}$ is bounded. From this, Lemma 4.4 and by taking subsequences if necessary, we see that there exist a constant $\alpha \in [0,\infty)$ and a vector field $\hat{\mathbf{a}} : B_3 \times \overline{\mathbb{K}} \times \mathbb{R}^n \to \mathbb{R}^n$ being continuous in the ξ variable such that $\alpha_k \to \alpha$ and $\hat{\mathbf{a}}_k(x,z,\xi) \to \hat{\mathbf{a}}(x,z,\xi)$ for a.e. $x \in B_3$ and for all $(z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n$. Moreover, (4.19) implies that, up to a subsequence, $d_{\mathbf{A}_k,\mathbf{a}_k}(x) \to 0$ for a.e. $x \in B_3$. Thus, we also have $\hat{\mathbf{A}}_k(x,z,\xi) \to \hat{\mathbf{a}}(x,z,\xi)$ for a.e. $x \in B_3$ and for all $(z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n$.

By using the pointwise convergence, it can be verified that $\hat{\mathbf{a}}$ satisfies conditions (1.2)–(1.4). We are going to derive a contradiction by proving the following claim.

Claim. There are subsequences $\{u^{k_m}\}$ and $\{v^{k_m}\}$ such that $u^{k_m} - v^{k_m} \to 0$ in $L^p(B_3)$ as $m \to \infty$.

Let us consider the case $\alpha > 0$ first. Then, thanks to (4.20), the sequence $\{u^k\}$ is bounded in $W^{1,p}(B_3)$. Likewise, by using (3.5) and Proposition 3.2 with $\mathbf{A}(x,z,\xi) \equiv \hat{\mathbf{a}}_k(x,\alpha_k z,\xi)$ and $\mathbf{F} \equiv 0$, we also have that the sequence $\{v^k\}$ is bounded in $W^{1,p}(B_3)$. Therefore there exist subsequences, still denoted by $\{u^k\}$ and $\{v^k\}$, and functions $u, v \in W^{1,p}(B_3)$ such that

$$\begin{cases} u^k \to u \text{ a.e. in } B_3, \quad u^k \to u \text{ strongly in } L^p(B_3), \quad \nabla u^k \rightharpoonup \nabla u \text{ weakly in } L^p(B_3), \\ v^k \to v \text{ a.e. in } B_3, \quad v^k \to v \text{ strongly in } L^p(B_3), \quad \nabla v^k \rightharpoonup \nabla v \text{ weakly in } L^p(B_3). \end{cases}$$

In particular, we have

$$u(x),\ v(x) \in \frac{1}{\alpha}\overline{\mathbb{K}} \ \text{ for a.e. } x \in B_3, \quad \text{and} \quad u = v \quad \text{on} \quad \partial B_3. \tag{4.24}$$

Also as the sequence $\left\{\hat{\mathbf{A}}_k(x,\alpha_k u^k,\nabla u^k)\right\}$ is bounded in $L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n)$, by taking a subsequence there exists $\zeta \in L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n)$ such that

$$\hat{\mathbf{A}}_k(x,\alpha_k u^k,\nabla u^k) \rightharpoonup \zeta \ \text{ weakly in } L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n).$$

But by applying Lemma 4.5 for $m_k(x) \rightsquigarrow \alpha_k u^k(x)$ and $m(x) \rightsquigarrow \alpha u(x)$, we obtain $\zeta \equiv \hat{\mathbf{a}}(x,\alpha u,\nabla u)$. That is,

$$\hat{\mathbf{A}}_k(x,\alpha_k u^k,\nabla u^k) \rightharpoonup \hat{\mathbf{a}}(x,\alpha u,\nabla u) \ \text{ weakly in } L^{\frac{p}{p-1}}(B_3;\mathbb{R}^n).$$

Therefore,

$$\lim_{k\to\infty}\int_{B_3}\langle\hat{\mathbf{A}}_k(x,\alpha_k u^k,\nabla u^k),\nabla\varphi\rangle dx = \int_{B_3}\langle\hat{\mathbf{a}}(x,\alpha u,\nabla u),\nabla\varphi\rangle dx \quad \text{for all} \quad \varphi \in C_0^\infty(B_3). \tag{4.25}$$

Thus by passing $k \to \infty$ for equation (4.23), one sees that u is a weak solution of the equation

$$\operatorname{div}\hat{\mathbf{a}}(x,\alpha u,\nabla u) = 0 \quad \text{in} \quad B_3. \tag{4.26}$$

Similarly, v is a weak solution of

$$\operatorname{div}\hat{\mathbf{a}}(x,\alpha v,\nabla v) = 0 \quad \text{in} \quad B_3.$$

Hence due to (4.24) and by the uniqueness of the weak solution to equation (4.26) as explained in Remark 2.2, we conclude that $u \equiv v$ in B_3. It follows that $u^k - v^k \to u - v = 0$ strongly in $L^p(B_3)$.

Now, consider the case $\alpha = 0$, that is, $\alpha_k \to 0$. Let $\overline{u}^k := \alpha_k u^k$, $\overline{v}^k := \alpha_k v^k$, $w^k := u^k - u^k_{B_3}$ and $h^k := v^k - u^k_{B_3}$, where $u^k_{B_3} := \fint_{B_3} u^k(x)\,dx$. Then $w^k \in W^{1,p}_{loc}(B_4)$ is a weak solution of

$$\operatorname{div}\hat{\mathbf{A}}_k(x,\overline{u}^k,\nabla w^k) = \operatorname{div}\mathbf{F}_k \quad \text{in} \quad B_4 \tag{4.27}$$

and $h^k \in W^{1,p}(B_3)$ is a weak solution of

$$\begin{cases} \operatorname{div}\hat{\mathbf{a}}_k(x,\overline{v}^k,\nabla h^k) &= 0 \quad \text{in} \quad B_3, \\ h^k &= w^k \quad \text{on} \quad \partial B_3. \end{cases} \tag{4.28}$$

By applying Proposition 3.2 for $w \rightsquigarrow v^k$, $\varphi \rightsquigarrow u^k$, $\mathbf{F} \equiv 0$ and using (4.20), we get

$$\int_{B_3}|\nabla v^k|^p\,dx \le C\int_{B_3}|\nabla u^k|^p\,dx \le C \quad \text{for all } k. \tag{4.29}$$

Consequently,

$$\int_{B_3} |\nabla u^k - \nabla v^k|^p \, dx \le 2^{p-1} \left[\int_{B_3} |\nabla u^k|^p \, dx + \int_{B_3} |\nabla v^k|^p \, dx \right] \le C$$

which together with the Sobolev's inequality gives

$$\int_{B_3} |u^k - v^k|^p \, dx \le C \int_{B_3} |\nabla (u^k - v^k)|^p \, dx \le C. \tag{4.30}$$

Notice that on one hand the Poincaré inequality gives

$$\int_{B_3} |w^k|^p \, dx = \int_{B_3} |u^k - u^k_{B_3}|^p \, dx \le C \int_{B_3} |\nabla u^k|^p \, dx \le C.$$

On the other hand, by employing the Poincaré inequality, (4.29) and (4.30) we obtain

$$\begin{aligned} \|h^k\|_{L^p(B_3)} &= \|v^k - u^k_{B_3}\|_{L^p(B_3)} \le \|v^k - v^k_{B_3}\|_{L^p(B_3)} + |B_3|^{\frac{1}{p}} \, |u^k_{B_3} - v^k_{B_3}| \\ &\le C\|\nabla v^k\|_{L^p(B_3)} + \|u^k - v^k\|_{L^p(B_3)} \le C. \end{aligned}$$

Therefore, $\{w^k\}$ and $\{h^k\}$ are bounded sequences in $W^{1,p}(B_3)$. Moreover, $\{\overline{v}^k\}$ is bounded in $W^{1,p}(B_3)$ owing to (4.29) and

$$\|v^k\|_{L^p(B_3)} \le \|u^k\|_{L^p(B_3)} + \|u^k - v^k\|_{L^p(B_3)} \le |B_4|^{\frac{1}{p}} \, \frac{M_0}{\alpha_k} + C.$$

Consequently there are subsequences, still denoted by $\{w^k\}$, $\{h^k\}$ and $\{\overline{v}^k\}$ and three functions $w, h, \overline{v} \in W^{1,p}(B_3)$ such that

$$\begin{cases} w^k \to w \text{ a.e. in } B_3, & w^k \to w \text{ strongly in } L^p(B_3), \quad \nabla w^k \rightharpoonup \nabla w \text{ weakly in } L^p(B_3), \\ h^k \to h \text{ a.e. in } B_3, & h^k \to h \text{ strongly in } L^p(B_3), \quad \nabla h^k \rightharpoonup \nabla h \text{ weakly in } L^p(B_3), \\ \overline{v}^k \to \overline{v} \text{ a.e. in } B_3, & \overline{v}^k \to \overline{v} \text{ strongly in } L^p(B_3), \quad \nabla \overline{v}^k \rightharpoonup \nabla \overline{v} \text{ weakly in } L^p(B_3). \end{cases}$$

Since $\nabla \overline{v}^k = \alpha_k \nabla v^k \to 0$ in $L^p(B_3)$ thanks to (4.29), we infer further that $\nabla \overline{v}^k \to \nabla \overline{v} \equiv 0$ strongly in $L^p(B_3)$. Thus, $\overline{v}$ is a constant function. As $\nabla \overline{u}^k = \alpha_k \nabla u^k \to 0$ in $L^p(B_3)$ and

$$\begin{aligned} \|\overline{u}^k - \overline{v}\|_{L^p(B_3)} &\le \|\overline{u}^k - \overline{v}^k\|_{L^p(B_3)} + \|\overline{v}^k - \overline{v}\|_{L^p(B_3)} \\ &= \alpha_k \|u^k - v^k\|_{L^p(B_3)} + \|\overline{v}^k - \overline{v}\|_{L^p(B_3)} \le C\alpha_k + \|\overline{v}^k - \overline{v}\|_{L^p(B_3)}, \end{aligned}$$

we also have $\overline{u}^k \to \overline{v}$ strongly in $W^{1,p}(B_3)$. By taking a further subsequence, we can assume that $\overline{u}^k(x) \to \overline{v}$ a.e. in B_3.

It follows from Lemma 4.5 for $m_k(x) \rightsquigarrow \overline{u}^k(x)$ and $m(x) \rightsquigarrow \overline{v}$ that

$$\hat{\mathbf{A}}_k(x, \overline{u}^k, \nabla w^k) \rightharpoonup \hat{\mathbf{a}}(x, \overline{v}, \nabla w) \ \text{ weakly in } L^{\frac{p}{p-1}}(B_3; \mathbb{R}^n)$$

up to a subsequence. Then as in (4.25), one gets for all $\varphi \in C_0^\infty(B_3)$ that

$$\lim_{k \to \infty} \int_{B_3} \langle \hat{\mathbf{A}}_k(x, \overline{u}^k, \nabla w^k), \nabla \varphi \rangle dx = \int_{B_3} \langle \hat{\mathbf{a}}(x, \overline{v}, \nabla w), \nabla \varphi \rangle dx.$$

Hence by passing to the limit in equation (4.27), we conclude that w is a weak solution of

$$\operatorname{div} \hat{\mathbf{a}}(x, \overline{v}, \nabla w) = 0 \ \text{ in } \ B_3.$$

Likewise, we deduce from (4.28) that h is a weak solution of

$$\begin{cases} \operatorname{div} \hat{\mathbf{a}}(x, \overline{v}, \nabla h) = 0 & \text{in } B_3, \\ h = w & \text{on } \partial B_3. \end{cases} \tag{4.31}$$

By the uniqueness of the weak solution to equation (4.31), we conclude that $h \equiv w$ in B_3. This gives, again, $u^k - v^k = w^k - h^k \to 0$ in $L^p(B_3)$ as $k \to \infty$. Therefore, we have proved the Claim which contradicts (4.21). Thus the proof of (4.18) is complete. □

5. Approximating gradients of solutions

Throughout this section, let $\omega : [0,\infty) \to [0,\infty)$ be the function defined by

$$\omega(r) = \begin{cases} r\Lambda & \text{if } \ 0 \le r \le 2, \\ 2\Lambda & \text{if } \ r > 2. \end{cases} \tag{5.1}$$

Notice that if $\mathbf{a}$ satisfies (1.3)–(1.4), then we obtain from the definition of ω that

$$|\mathbf{a}(x,z_1,\xi) - \mathbf{a}(x,z_2,\xi)| \le \omega(|z_1 - z_2|)\,|\xi|^{p-1} \quad \forall z_1, z_2 \in \overline{\mathbb{K}}. \tag{5.2}$$

Our aim is to approximate ∇u by a good gradient in L^p norm, and the following lemma is the starting point for that purpose.

Lemma 5.1. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.3), $\mathbf{a} \in \mathbb{G}_{B_3}(\eta)$, $M_0 > 0$, $\lambda > 0$, *and* $0 < \theta \le 1$. *Let* $u \in W^{1,p}_{loc}(B_4)$ *be a weak solution of* (4.15) *with* $⨍_{B_4} |\nabla u|^p\,dx \le 1$ *and* $\|\mathbf{F}\|_{L^{\frac{p}{p-1}}(B_4)} \le 1$. *Then for any weak solution* $v \in W^{1,p}(B_3)$ *of* (4.17) *satisfying* $\|v\|_{L^\infty(B_3)} \le \frac{M_0}{\lambda\theta}$, *we have:*

(i) *If* $p \ge 2$, *then*

$$\int_{B_2} |\nabla u - \nabla v|^p dx \le C\left\{\int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|) + d_{\mathbf{A},\mathbf{a}}(x)\Big]^{\frac{p}{p-1}} dx + \int_{B_{\frac{5}{2}}} |\mathbf{F}|^{\frac{p}{p-1}} dx\right\} + C\|u - v\|_{L^p(B_{\frac{5}{2}})}.$$

(ii) *If* $1 < p < 2$, *then*

$$\begin{aligned}\int_{B_2} |\nabla u - \nabla v|^p dx \le \frac{C}{\sigma^{\frac{p}{p-1}}}\left\{\int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|) + d_{\mathbf{A},\mathbf{a}}(x)\Big]^{\frac{p}{p-1}} dx + \int_{B_{\frac{5}{2}}} |\mathbf{F}|^{\frac{p}{p-1}} dx\right\} \\ + C\sigma^{\frac{p}{2-p}} + \frac{C}{\sigma}\|u - v\|_{L^p(B_{\frac{5}{2}})} \qquad \textit{for every } \sigma > 0 \textit{ small.}\end{aligned}$$

Here the constant $C > 0$ *depends only on* p, n, η, Λ, $\mathbb{K}$ *and* M_0.

Proof. Observe that if we let $\bar{v}(y) := \lambda\theta v(y/\theta)$, then $\|\bar{v}\|_{L^\infty(B_{3\theta})} \le M_0$ and $\bar{v}$ is a weak solution of $\operatorname{div} \mathbf{a}(y, \bar{v}, \nabla\bar{v}) = 0$ in $B_{3\theta}$. Thus assumption **(H3)** about interior $W^{1,\infty}$-estimates gives

$$\|\nabla\bar{v}\|^p_{L^\infty(B_{\frac{5\theta}{2}})} \le C(p,n,\eta,\Lambda,\mathbb{K},M_0) ⨍_{B_{3\theta}} |\nabla\bar{v}|^p\,dx. \tag{5.3}$$

On the other hand, it follows from Proposition 3.2 and the assumptions that

$$\|\nabla v\|_{L^p(B_3)} \le C\|\nabla u\|_{L^p(B_3)} \le C(p,n,\Lambda).$$

Therefore, we have from (5.3) by rescaling back that

$$\|\nabla v\|^p_{L^\infty(B_{\frac{5}{2}})} \le C(p,n,\eta,\Lambda,\mathbb{K},M_0) ⨍_{B_3} |\nabla v|^p\,dx \le C(p,n,\eta,\Lambda,\mathbb{K},M_0). \tag{5.4}$$

Next for convenience, set

$$\hat{\mathbf{A}}(x,z,\xi) := \frac{\mathbf{A}(x,z,\lambda\xi)}{\lambda^{p-1}} \quad \text{and} \quad \hat{\mathbf{a}}(x,z,\xi) := \frac{\mathbf{a}(x,z,\lambda\xi)}{\lambda^{p-1}}.$$

Let φ be the standard nonnegative cut-off function which is 1 on B_2 and $\mathrm{supp}(\varphi) \subset B_{\frac{5}{2}}$. Then by using $\varphi^p(u-v)$ as a test function in the equations for u and v, we have

$$\begin{aligned}
&\int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{A}}(x, \lambda\theta u, \nabla u), \nabla u - \nabla v\rangle \varphi^p \, dx = -p \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{A}}(x, \lambda\theta u, \nabla u), \nabla\varphi\rangle (u-v)\varphi^{p-1} \, dx \\
&+ p \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{a}}(x, \lambda\theta v, \nabla v), \nabla\varphi\rangle (u-v)\varphi^{p-1} \, dx + \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{a}}(x, \lambda\theta v, \nabla v), \nabla u - \nabla v\rangle \varphi^p dx \\
&+ \int_{B_{\frac{5}{2}}} \langle \mathbf{F}, \nabla u - \nabla v\rangle \varphi^p dx + p \int_{B_{\frac{5}{2}}} \langle \mathbf{F}, \nabla\varphi\rangle (u-v)\varphi^{p-1} dx.
\end{aligned}$$

This gives

$$\begin{aligned}
I &:= \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{A}}(x, \lambda\theta u, \nabla u) - \hat{\mathbf{A}}(x, \lambda\theta u, \nabla v), \nabla u - \nabla v\rangle \varphi^p \, dx \\
&= p \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{a}}(x, \lambda\theta v, \nabla v) - \hat{\mathbf{A}}(x, \lambda\theta u, \nabla u), \nabla\varphi\rangle (u-v)\varphi^{p-1} \, dx \\
&\quad + \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{a}}(x, \lambda\theta v, \nabla v) - \hat{\mathbf{a}}(x, \lambda\theta u, \nabla v), \nabla u - \nabla v\rangle \varphi^p \, dx \\
&\quad - \int_{B_{\frac{5}{2}}} \langle \hat{\mathbf{A}}(x, \lambda\theta u, \nabla v) - \hat{\mathbf{a}}(x, \lambda\theta u, \nabla v), \nabla u - \nabla v\rangle \varphi^p \, dx \\
&\quad + \int_{B_{\frac{5}{2}}} \langle \mathbf{F}, \nabla u - \nabla v\rangle \varphi^p \, dx + p \int_{B_{\frac{5}{2}}} \langle \mathbf{F}, \nabla\varphi\rangle (u-v)\varphi^{p-1} dx.
\end{aligned}$$

We deduce from this, the structural conditions, (5.2) and (5.4) that

$$\begin{aligned}
I &\le p\Lambda \int_{B_{\frac{5}{2}}} (|\nabla v|^{p-1} + |\nabla u|^{p-1}) |\nabla\varphi| |u-v| \varphi^{p-1} \, dx \\
&\quad + \Lambda \int_{B_{\frac{5}{2}}} \frac{\omega(|\lambda\theta(u-v)|)}{\lambda^{p-1}} |\nabla(\lambda v)|^{p-1} |\nabla u - \nabla v| \varphi^p \, dx + \int_{B_{\frac{5}{2}}} d_{\mathbf{A},\mathbf{a}}(x) |\nabla v|^{p-1} |\nabla u - \nabla v| \varphi^p \, dx \\
&\quad + \int_{B_{\frac{5}{2}}} |\mathbf{F}| |\nabla u - \nabla v| \varphi^p \, dx + p \int_{B_{\frac{5}{2}}} |\mathbf{F}| |\nabla\varphi| |u-v| \varphi^{p-1} dx \\
&\le C \int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|)| + d_{\mathbf{A},\mathbf{a}}(x)\Big] |\nabla u - \nabla v| \varphi^p \, dx + \int_{B_{\frac{5}{2}}} |\mathbf{F}| |\nabla u - \nabla v| \varphi^p \, dx \\
&\quad + C\Big(\|\nabla v\|_{L^p(B_{\frac{5}{2}})} + \|\nabla u\|_{L^p(B_{\frac{5}{2}})} + \|\mathbf{F}\|_{L^{\frac{p}{p-1}}(B_{\frac{5}{2}})}\Big) \|u-v\|_{L^p(B_{\frac{5}{2}})} \\
&\le \frac{2\sigma}{p} \int_{B_{\frac{5}{2}}} |\nabla u - \nabla v|^p \varphi^p \, dx + C \frac{p-1}{p\sigma^{\frac{1}{p-1}}} \int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|) + d_{\mathbf{A},\mathbf{a}}(x)\Big]^{\frac{p}{p-1}} dx \\
&\quad + \frac{p-1}{p\sigma^{\frac{1}{p-1}}} \int_{B_{\frac{5}{2}}} |\mathbf{F}|^{\frac{p}{p-1}} dx + C\|u-v\|_{L^p(B_{\frac{5}{2}})}
\end{aligned} \tag{5.5}$$

for any $\sigma > 0$.

Now if $p \geq 2$, then (1.2) implies $\Lambda^{-1}\int_{B_{\frac{5}{2}}} |\nabla u - \nabla v|^p \varphi^p dx \leq I$. Hence by combining with (5.5) and choosing $\sigma > 0$ sufficiently small, we obtain

$$\int_{B_{\frac{5}{2}}} |\nabla u - \nabla v|^p \varphi^p dx \leq C\left\{\int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|) + d_{\mathbf{A},\mathbf{a}}(x)\Big]^{\frac{p}{p-1}} dx + \int_{B_{\frac{5}{2}}} |\mathbf{F}|^{\frac{p}{p-1}} dx\right\} + C\|u-v\|_{L^p(B_{\frac{5}{2}})}$$

giving (i). On the other hand, if $1 < p < 2$ then Lemma 3.1 yields

$$c\tau^{\frac{2-p}{p}} \int_{B_{\frac{5}{2}}} |\nabla u - \nabla v|^p \varphi^p dx - \tau^{\frac{2}{p}} \int_{B_{\frac{5}{2}}} |\nabla u|^p \varphi^p dx \leq I$$

for all $\tau > 0$ small. By combining this with the assumptions and (5.5) we deduce that

$$(\tau^{\frac{2-p}{p}} - \sigma) \int_{B_{\frac{5}{2}}} |\nabla u - \nabla v|^p \varphi^p dx \leq \frac{C}{\sigma^{\frac{1}{p-1}}}\left\{\int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|) + d_{\mathbf{A},\mathbf{a}}(x)\Big]^{\frac{p}{p-1}} dx + \int_{B_{\frac{5}{2}}} |\mathbf{F}|^{\frac{p}{p-1}} dx\right\}$$
$$+ C\tau^{\frac{2}{p}} + C\|u-v\|_{L^p(B_{\frac{5}{2}})}.$$

It then follows by taking $\tau^{\frac{2-p}{p}} = 2\sigma$ that

$$\int_{B_{\frac{5}{2}}} |\nabla u - \nabla v|^p \varphi^p dx \leq \frac{C}{\sigma^{\frac{p}{p-1}}}\left\{\int_{B_{\frac{5}{2}}} \Big[\omega(|\lambda\theta(u-v)|) + d_{\mathbf{A},\mathbf{a}}(x)\Big]^{\frac{p}{p-1}} dx + \int_{B_{\frac{5}{2}}} |\mathbf{F}|^{\frac{p}{p-1}} dx\right\}$$
$$+ C\sigma^{\frac{p}{2-p}} + \frac{C}{\sigma}\|u-v\|_{L^p(B_{\frac{5}{2}})}$$

for every $\sigma > 0$ small. □

As a consequence of Lemma 4.6 and Lemma 5.1, we obtain:

Corollary 5.2. *Let* $\mathbf{A}$ *satisfy* (1.2)–(1.4), $\mathbf{a} \in \mathbb{G}_{B_3}(\eta)$, *and* $M_0 > 0$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *depending only on* ε, Λ, p, η, n, $\mathbb{K}$ *and* M_0 *such that: if* $\lambda > 0$, $0 < \theta \leq 1$,

$$\fint_{B_3} d_{\mathbf{A},\mathbf{a}}(x)\, dx \leq \delta, \qquad \fint_{B_4} |\mathbf{F}|^{\frac{p}{p-1}}\, dx \leq \delta,$$

and $u \in W^{1,p}_{loc}(B_4)$ *is a weak solution of* (4.15) *satisfying*

$$\Big(\fint_{B_4} |u|^p\, dx\Big)^{\frac{1}{p}} \leq \frac{M_0}{\lambda\theta} \quad \text{and} \quad \fint_{B_4} |\nabla u|^p\, dx \leq 1,$$

and $v \in W^{1,p}(B_3)$ *is a weak solution of* (4.17) *with* $\|v\|_{L^\infty(B_3)} \leq \frac{M_0}{\lambda\theta}$, *then*

$$\int_{B_2} |\nabla u - \nabla v|^p\, dx \leq \varepsilon^p.$$

Proof. We will present the proof only for the case $1 < p < 2$ as the case $p \geq 2$ is simpler. Let $\varepsilon > 0$ be arbitrary. By Lemma 3.4 and the assumptions, we have

$$\int_{B_3} |\nabla u - \nabla v|^p\, dx \leq C(p,n,\Lambda) \int_{B_4} (|u|^p + |\mathbf{F}|^{\frac{p}{p-1}})\, dx \leq C^*\Big[(\frac{M_0}{\lambda\theta})^p + \delta\Big].$$

Therefore, the conclusion of the lemma follows if $\lambda\theta > \frac{(2C^*)^{1/p} M_0}{\varepsilon}$. Thus, it remains to consider the case

$$\lambda\theta \leq \frac{(2C^*)^{\frac{1}{p}} M_0}{\varepsilon}. \tag{5.6}$$

Now from Lemma 5.1(ii) and the boundedness of $d_{\mathbf{A},\mathbf{a}}(x)$ we get

$$\int_{B_2} |\nabla u - \nabla v|^p dx \leq \frac{C}{\sigma^{\frac{p}{p-1}}} \left\{ \int_{B_{\frac{5}{2}}} \omega(|\lambda\theta(u-v)|)^{\frac{p}{p-1}}\, dx + \int_{B_{\frac{5}{2}}} (d_{\mathbf{A},\mathbf{a}} + |\mathbf{F}|^{\frac{p}{p-1}})\, dx \right\} + C\sigma^{\frac{p}{2-p}} + \frac{C}{\sigma}\|u-v\|_{L^p(B_{\frac{5}{2}})} \qquad \text{for all small} \quad \sigma > 0. \tag{5.7}$$

Notice that for any $\tau > 0$ small, from the definition of ω in (5.1) we have

$$\omega(s) \leq \tau + \frac{2\,\Lambda^p}{\tau^{p-1}} s^{p-1} \quad \forall s \geq 0.$$

Therefore by combining with (5.6), one easily sees that

$$\omega(|\lambda\theta(u-v)|)^{\frac{p}{p-1}} \leq \tau + \frac{C}{\tau^p}(\lambda\theta)^p |u-v|^p \leq \tau + \frac{C}{\tau^p \varepsilon^p}|u-v|^p \quad \forall \tau > 0 \text{ small.}$$

Hence by first selecting $\sigma = \sigma(\epsilon, p) > 0$ small such that $C\sigma^{\frac{p}{2-p}} \leq \epsilon^p/5$ and then choosing $\tau > 0$ such that $C\tau/\sigma^{\frac{p}{p-1}} \leq \epsilon^p/5$, we conclude from (5.7) that

$$\int_{B_2} |\nabla u - \nabla v|^p dx \leq 2\frac{\epsilon^p}{5} + \frac{C}{\tau^p \varepsilon^p \sigma^{\frac{p}{p-1}}}\|u-v\|^p_{L^p(B_{\frac{5}{2}})} + \frac{C\delta}{\sigma^{\frac{p}{p-1}}} + \frac{C}{\sigma}\|u-v\|_{L^p(B_{\frac{5}{2}})}. \tag{5.8}$$

Next let us pick $\epsilon' > 0$ small so that

$$\frac{C}{\tau^p \varepsilon^p \sigma^{\frac{p}{p-1}}}(\epsilon')^p \leq \frac{\epsilon^p}{5} \quad \text{and} \quad \frac{C}{\sigma}\epsilon' \leq \frac{\epsilon^p}{5}.$$

Then by Lemma 4.6 there exists $\delta' > 0$ such that $\|u-v\|_{L^p(B_3)} \leq \epsilon'$ if $\fint_{B_3} d_{\mathbf{A},\mathbf{a}}(x)\, dx \leq \delta'$ and $\fint_{B_4} |\mathbf{F}|^{\frac{p}{p-1}}\, dx \leq \delta'$. Thus, by taking $\delta := \min\{\delta', \frac{\epsilon^p \sigma^{\frac{p}{p-1}}}{5C}\}$ we obtain the corollary from (5.8). □

The next result is a localized version of Corollary 5.2.

Lemma 5.3. *Let* $\mathbf{A}$ *satisfy* (1.2)–(1.4)*, and* $M_0 > 0$*. For any* $\varepsilon > 0$*, there exists* $\delta > 0$ *depending only on* ε, Λ, p, η, n, $\mathbb{K}$ *and* M_0 *such that: if* $\lambda > 0$, $0 < \theta \leq 1$, $0 < r \leq 1$,

$$dist\Big(\mathbf{A}, \mathbb{G}_{B_{3r}}\Big) < \delta \quad \text{and} \quad \fint_{B_{4r}} |\mathbf{F}|^{\frac{p}{p-1}}\, dx \leq \delta, \tag{5.9}$$

and $u \in W^{1,p}_{loc}(B_{4r})$ *is a weak solution of* $div\left[\frac{\mathbf{A}(x,\lambda\theta u,\lambda\nabla u)}{\lambda^{p-1}}\right] = div\,\mathbf{F}$ *in* B_{4r} *satisfying*

$$\|u\|_{L^\infty(B_{4r})} \leq \frac{M_0}{\lambda\theta} \quad \text{and} \quad \fint_{B_{4r}} |\nabla u|^p\, dx \leq 1,$$

then

$$\fint_{B_{2r}} |\nabla u - \nabla v|^p\, dx \leq \varepsilon^p \tag{5.10}$$

for some function $v \in W^{1,p}(B_{3r})$ *with*

$$\|\nabla v\|^p_{L^\infty(B_{\frac{3r}{2}})} \leq C(p,n,\eta,\Lambda,\mathbb{K},M_0) \fint_{B_{2r}} |\nabla v|^p\, dx. \tag{5.11}$$

Proof. It follows from Definition 2.1 and the first condition in (5.9) that there exists a vector field $\mathbf{a}' \in \mathbb{G}_{B_3}(\eta)$ such that

$$\fint_{B_{3r}} d_{\mathbf{A},\mathbf{a}}(y)\, dy \leq \delta \quad \text{with} \quad \mathbf{a}(y,z,\xi) := \mathbf{a}'(\frac{y}{r}, z, \xi).$$

Define
$$\mathbf{A}'(x,z,\xi)=\mathbf{A}(rx,z,\xi),\quad \mathbf{F}'(x)=\mathbf{F}(rx),\quad u'(x)=\frac{u(rx)}{r}.$$
Let $\theta':=\theta r\in(0,1]$. Then $u'\in W^{1,p}_{loc}(B_4)$ is a weak solution of
$$\operatorname{div}\Big[\frac{\mathbf{A}'(x,\lambda\theta' u',\lambda\nabla u')}{\lambda^{p-1}}\Big]=\operatorname{div}\mathbf{F}'\quad\text{in}\quad B_4.$$
Notice that $\|u'\|_{L^\infty(B_4)}\le\frac{M_0}{\lambda\theta'}$ and $d_{\mathbf{A}',\mathbf{a}'}(x)=d_{\mathbf{A},\mathbf{a}}(rx)$. Thus we also have
$$\Big(\fint_{B_4}|u'(x)|^p\,dx\Big)^{\frac1p}=\frac1r\Big(\fint_{B_{4r}}|u(y)|^p\,dy\Big)^{\frac1p}\le\frac{M_0}{\lambda\theta'},\qquad \fint_{B_4}|\nabla u'(x)|^p\,dx=\fint_{B_{4r}}|\nabla u(y)|^p\,dy\le 1,$$
$$\fint_{B_3}d_{\mathbf{A}',\mathbf{a}'}(x)\,dx=\fint_{B_{3r}}d_{\mathbf{A},\mathbf{a}}(y)\,dy,\qquad \fint_{B_4}|\mathbf{F}'(x)|^{\frac{p}{p-1}}\,dx=\fint_{B_{4r}}|\mathbf{F}(y)|^{\frac{p}{p-1}}\,dy.$$
Therefore, given any $\varepsilon>0$, by Corollary 5.2 there exists a constant $\delta=\delta(\varepsilon,\Lambda,p,\eta,n,\mathbb{K},M_0)>0$ such that if condition (5.9) for $\mathbf{A}$ and $\mathbf{F}$ is satisfied then we have
$$\int_{B_2}|\nabla u'(x)-\nabla v'(x)|^p\,dx\le 2^n\omega_n\varepsilon^p,\tag{5.12}$$
where $v'\in W^{1,p}(B_3)$ is a weak solution of
$$\begin{cases}\operatorname{div}\Big[\frac{\mathbf{a}'(x,\lambda\theta' v',\lambda\nabla v')}{\lambda^{p-1}}\Big] &= 0 \quad\text{in}\quad B_3,\\ v' &= u' \quad\text{on}\quad \partial B_3\end{cases}$$
satisfying $\|v'\|_{L^\infty(B_3)}\le\frac{M_0}{\lambda\theta'}$. Notice that the existence of such weak solution v' to the above Dirichlet problem is guaranteed by Remark 2.2. Now let $v(x):=rv'(x/r)$ for $x\in B_{3r}$. Then by changing variables, we obtain the desired estimate (5.10) from (5.12).

It remains to show (5.11). Define $\bar v(y)=\lambda\theta' v'(y/\theta')$. Then $\|\bar v\|_{L^\infty(B_{3\theta'})}\le M_0$ and $\bar v$ is a weak solution of
$$\operatorname{div}\mathbf{a}'(y,\bar v,\nabla\bar v)=0\quad\text{in}\quad B_{3\theta'}.$$
Since $0<\theta'\le 1$ and $\mathbf{a}'\in\mathbb{G}_{B_3}(\eta)$, the assumption **(H3)** about interior $W^{1,\infty}$-estimates gives
$$\|\nabla\bar v\|^p_{L^\infty(B_{\frac{3\theta'}{2}})}\le C(p,n,\eta,\Lambda,\mathbb{K},M_0)\fint_{B_{2\theta'}}|\nabla\bar v|^p\,dx.$$
This yields (5.11) owing to $\bar v(y)=\lambda\theta v(y/\theta)$ and $\theta'/\theta=r$.

□

Remark 5.4. *Since the class of our equations is invariant under the transformation $x\mapsto x+y$, Lemma 5.3 still holds true if B_r is replaced by $B_r(y)$.*

6. Density and gradient estimates

We will derive interior $W^{1,q}$-estimates for solution u of (2.3) by estimating the distribution functions of the maximal function of $|\nabla u|^p$. The precise maximal operators will be used are:

Definition 6.1. *The Hardy–Littlewood maximal function of a function $f\in L^1_{loc}(\mathbb{R}^n)$ is defined by*
$$(\mathcal{M}f)(x)=\sup_{\rho>0}\fint_{B_\rho(x)}|f(y)|\,dy.$$
In case U is a region in $\mathbb{R}^n$ and $f\in L^1(U)$, then we denote $\mathcal{M}_U f=\mathcal{M}(\chi_U f)$.

The next result gives a density estimate for the distribution of $\mathcal{M}_{B_5}(|\nabla u|^p)$. It roughly says that if the maximal function $\mathcal{M}_{B_5}(|\nabla u|^p)$ is bounded at one point in $B_r(y)$ then this property can be propagated for all points in $B_r(y)$ except on a set of small measure.

Lemma 6.2. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4), $\mathbf{F} \in L^{\frac{p}{p-1}}(B_6;\mathbb{R}^n)$, *and* $M_0 > 0$. *There exists a constant* $N > 1$ *depending only on* p, n, η, Λ, $\mathbb{K}$ *and* M_0 *such that for any* $\varepsilon > 0$, *we can find* $\delta = \delta(\varepsilon, \Lambda, p, \eta, n, \mathbb{K}, M_0) > 0$ *satisfying: if* $\lambda > 0$, $0 < \theta \leq 1$,

$$\sup_{0<\rho\leq 3}\sup_{y\in B_1} dist\Big(\mathbf{A}, \mathbb{G}_{B_\rho(y)}(\eta)\Big) < \delta,$$

then for any weak solution u *of* (2.3) *with* $\|u\|_{L^\infty(B_5)} \leq \frac{M_0}{\lambda\theta}$, *and for any* $y \in B_1$, $0 < r \leq 1$ *with*

$$B_r(y) \cap B_1 \cap \{B_5 : \mathcal{M}_{B_5}(|\nabla u|^p) \leq 1\} \cap \{B_5 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) \leq \delta\} \neq \emptyset, \tag{6.1}$$

we have

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\} \cap B_r(y)\Big| \leq \varepsilon |B_r(y)|.$$

Proof. By condition (6.1), there exists a point $x_0 \in B_r(y) \cap B_1$ such that

$$\mathcal{M}_{B_5}(|\nabla u|^p)(x_0) \leq 1 \quad \text{and} \quad \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}})(x_0) \leq \delta. \tag{6.2}$$

Since $B_{4r}(y) \subset B_{5r}(x_0) \cap B_5$, it follows from (6.2) that

$$\begin{aligned}
\fint_{B_{4r}(y)} |\nabla u|^p\,dx &\leq \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|}\frac{1}{|B_{5r}(x_0)|}\int_{B_{5r}(x_0)\cap B_5} |\nabla u|^p\,dx \leq \Big(\frac{5}{4}\Big)^n,\\
\fint_{B_{4r}(y)} |\mathbf{F}|^{\frac{p}{p-1}}\,dx &\leq \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|}\frac{1}{|B_{5r}(x_0)|}\int_{B_{5r}(x_0)\cap B_5} |\mathbf{F}|^{\frac{p}{p-1}}\,dx \leq \Big(\frac{5}{4}\Big)^n\delta.
\end{aligned}$$

Therefore, we can use Lemma 5.3 and Remark 5.4 to obtain

$$\fint_{B_{2r}(y)} |\nabla u - \nabla v|^p\,dx \leq \gamma^p, \tag{6.3}$$

where $v \in W^{1,p}(B_{3r}(y))$ is some function satisfying

$$\|\nabla v\|^p_{L^\infty(B_{\frac{3r}{2}}(y))} \leq C(p, n, \eta, \Lambda, \mathbb{K}, M_0)\fint_{B_{2r}(y)} |\nabla v|^p\,dx. \tag{6.4}$$

Here $\delta = \delta(\gamma, \Lambda, p, \eta, n, \mathbb{K}, M_0) > 0$ with $\gamma \in (0,1)$ being determined later. By using (6.4) together with (6.3) and (6.2), we get

$$\|\nabla v\|^p_{L^\infty(B_{\frac{3r}{2}}(y))} \leq 2^{p-1}C\left(\fint_{B_{2r}(y)} |\nabla u - \nabla v|^p\,dx + \fint_{B_{2r}(y)} |\nabla u|^p\,dx\right) \leq C_*(\gamma^p + 1), \tag{6.5}$$

where $C_* = C_*(p, n, \eta, \Lambda, \mathbb{K}, M_0)$. We claim that (6.2), (6.3) and (6.5) yield

$$\{B_r(y) : \mathcal{M}_{B_{2r}(y)}(|\nabla u - \nabla v|^p) \leq C_*\} \subset \{B_r(y) : \mathcal{M}_{B_5}(|\nabla u|^p) \leq N\} \tag{6.6}$$

with $N := \max\{2^{p+1}C_*, 5^n\}$. Indeed, let x be a point in the set on the left hand side of (6.6), and consider $B_\rho(x)$. If $\rho \leq r/2$, then $B_\rho(x) \subset B_{3r/2}(y) \subset B_3$ and hence

$$\begin{aligned}
\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)\cap B_5} |\nabla u|^p\,dx &\leq \frac{2^{p-1}}{|B_\rho(x)|}\Big[\int_{B_\rho(x)\cap B_5} |\nabla u - \nabla v|^p\,dx + \int_{B_\rho(x)\cap B_5} |\nabla v|^p\,dx\Big]\\
&\leq 2^{p-1}\Big[\mathcal{M}_{B_{2r}(y)}(|\nabla u - \nabla v|^p)(x) + \|\nabla v\|^p_{L^\infty(B_{\frac{3r}{2}}(y))}\Big]\\
&\leq 2^{p-1}C_*(\gamma^p + 2) \leq 2^{p+1}C_*.
\end{aligned}$$

On the other hand if $\rho > r/2$, then $B_\rho(x) \subset B_{5\rho}(x_0)$. This and the first inequality in (6.2) imply that

$$\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)\cap B_5} |\nabla u|^p\,dx \leq \frac{5^n}{|B_{5\rho}(x_0)|}\int_{B_{5\rho}(x_0)\cap B_5} |\nabla u|^p\,dx \leq 5^n.$$

Therefore, $\mathcal{M}_{B_5}(|\nabla u|^p)(x) \leq N$ and the claim (6.6) is proved. Note that (6.6) is equivalent to

$$\{B_r(y) : \mathcal{M}_{B_5}(|\nabla u|^p) > N\} \subset \{B_r(y) : \mathcal{M}_{B_{2r}(y)}(|\nabla u - \nabla v|^p) > C_*\}.$$

It follows from this, the weak type $1-1$ estimate and (6.3) that

$$\begin{aligned}\Big|\{B_r(y) : \mathcal{M}_{B_5}(|\nabla u|^p) > N\}\Big| &\leq \Big|\{B_r(y) : \mathcal{M}_{B_{2r}(y)}(|\nabla u - \nabla v|^p) > C_*\}\Big| \\ &\leq C \int_{B_{2r}(y)} |\nabla u - \nabla v|^p \, dx \leq C' \gamma^p \, |B_r(y)|,\end{aligned}$$

where $C' > 0$ depends only on p, n, η, Λ, $\mathbb{K}$ and M_0. By choosing $\gamma = \sqrt[p]{\frac{\varepsilon}{C'}}$, we obtain the desired result. □

In view of Lemma 6.2, we can apply the variation of the Vitali covering lemma given by [36, Theorem 3] (see also [6, Lemma 1.2]) for

$$C = \{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\} \quad \text{and} \quad D = \{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > 1\} \cup \{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta\}$$

to obtain:

Lemma 6.3. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4), $\mathbf{F} \in L^{\frac{p}{p-1}}(B_6; \mathbb{R}^n)$, *and* $M_0 > 0$. *There exists a constant* $N > 1$ *depending only on* p, n, η, Λ, $\mathbb{K}$ *and* M_0 *such that for any* $\varepsilon > 0$, *we can find* $\delta = \delta(\varepsilon, \Lambda, p, \eta, n, \mathbb{K}, M_0) > 0$ *satisfying: if* $\lambda > 0$, $0 < \theta \leq 1$,

$$\sup_{0<\rho\leq 3} \sup_{y \in B_1} dist\Big(\mathbf{A}, \mathbb{G}_{B_\rho(y)}(\eta)\Big) < \delta,$$

then for any weak solution $u \in W^{1,p}_{loc}(B_6)$ *of* (2.3) *satisfying*

$$\|u\|_{L^\infty(B_5)} \leq \frac{M_0}{\lambda\theta} \quad \text{and} \quad \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\}\Big| \leq \varepsilon |B_1|,$$

we have

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\}\Big| \leq 20^n \varepsilon \Big(\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > 1\}\Big| + \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta\}\Big|\Big).$$

6.1. **Interior gradient estimates in Lebesgue spaces.** We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $N > 1$ be as in Lemma 6.3, and let $q_1 = q/p > 1$. We choose $\varepsilon = \varepsilon(p, q, n, \eta, \Lambda, \mathbb{K}, M_0) > 0$ be such that

$$\varepsilon_1 \stackrel{\text{def}}{=} 20^n \varepsilon = \frac{1}{2N^{q_1}},$$

and let $\delta = \delta(p, q, n, \Lambda, \eta, \mathbb{K}, M_0)$ be the corresponding constant given by Lemma 6.3.

Assuming for a moment that u satisfies

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\}\Big| \leq \varepsilon |B_1|. \tag{6.7}$$

Then it follows from Lemma 6.3 that

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\}\Big| \leq \varepsilon_1 \left(\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > 1\}\Big| + \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta\}\Big|\right). \tag{6.8}$$

Let us iterate this estimate by considering

$$u_1(x) = \frac{u(x)}{N^{\frac{1}{p}}}, \quad \mathbf{F}_1(x) = \frac{\mathbf{F}(x)}{N^{\frac{p-1}{p}}} \quad \text{and} \quad \lambda_1 = N^{\frac{1}{p}} \lambda.$$

It is clear that $\|u_1\|_{L^\infty(B_5)} \leq \frac{M_0}{\lambda_1 \theta}$ and $u_1 \in W^{1,p}_{loc}(B_6)$ is a weak solution of

$$\operatorname{div}\Big[\frac{\mathbf{A}(x, \lambda_1 \theta u_1, \lambda_1 \nabla u_1)}{\lambda_1^{p-1}}\Big] = \operatorname{div} \mathbf{F}_1 \quad \text{in} \quad B_6.$$

Moreover, thanks to (6.7) we have

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u_1|^p) > N\}\Big| = \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^2\}\Big| \le \varepsilon |B_1|.$$

Therefore, by applying Lemma 6.3 to u_1 we obtain

$$\begin{aligned}\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u_1|^p) > N\}\Big| &\le \varepsilon_1 \Big(\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u_1|^p) > 1\}\Big| + \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}_1|^{\frac{p}{p-1}}) > \delta\}\Big|\Big)\\ &= \varepsilon_1 \Big(\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N\}\Big| + \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N\}\Big|\Big).\end{aligned}$$

We infer from this and (6.8) that

$$\begin{aligned}\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^2\}\Big| &\le \varepsilon_1^2 \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > 1\}\Big|\\ &+ \varepsilon_1^2 \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta\}\Big| + \varepsilon_1 \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N\}\Big|.\end{aligned} \tag{6.9}$$

Next, let

$$u_2(x) = \frac{u(x)}{N^{\frac{2}{p}}}, \quad \mathbf{F}_2(x) = \frac{\mathbf{F}(x)}{N^{\frac{2(p-1)}{p}}} \quad \text{and} \quad \lambda_2 = N^{\frac{2}{p}} \lambda.$$

Then u_2 is a weak solution of

$$\operatorname{div}\Big[\frac{\mathbf{A}(x, \lambda_2 \theta u_2, \lambda_2 \nabla u_2)}{\lambda_2^{p-1}}\Big] = \operatorname{div} \mathbf{F}_2 \quad \text{in} \quad B_6$$

satisfying

$$\|u_2\|_{L^\infty(B_5)} \le \frac{M_0}{\lambda_2 \theta} \quad \text{and} \quad \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u_2|^p) > N\}\Big| = \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^3\}\Big| \le \varepsilon |B_1|.$$

Hence by applying Lemma 6.3 to u_2 we get

$$\begin{aligned}\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u_2|^p) > N\}\Big| &\le \varepsilon_1 \Big(\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u_2|^p) > 1\}\Big| + \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}_2|^{\frac{p}{p-1}}) > \delta\}\Big|\Big)\\ &= \varepsilon_1 \Big(\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^2\}\Big| + \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^2\}\Big|\Big).\end{aligned}$$

This together with (6.9) gives

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^3\}\Big| \le \varepsilon_1^3 \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > 1\}\Big| + \sum_{i=1}^{3} \varepsilon_1^i \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{3-i}\}\Big|.$$

By repeating the iteration, we then conclude that

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^k\}\Big| \le \varepsilon_1^k \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > 1\}\Big| + \sum_{i=1}^{k} \varepsilon_1^i \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{k-i}\}\Big| \tag{6.10}$$

for all $k = 1, 2, \ldots$ This together with

$$\begin{aligned}&\int_{B_1} \mathcal{M}_{B_5}(|\nabla u|^p)^{q_1}\, dx = q_1 \int_0^\infty t^{q_1-1} \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > t\}\Big|\, dt\\ &= q_1 \int_0^N t^{q_1-1} \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > t\}\Big|\, dt + q_1 \sum_{k=1}^{\infty} \int_{N^k}^{N^{k+1}} t^{q_1-1} \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > t\}\Big|\, dt\\ &\le N^{q_1} |B_1| + (N^{q_1} - 1) \sum_{k=1}^{\infty} N^{q_1 k} \Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^k\}\Big|\end{aligned}$$

gives

$$\int_{B_1} \mathcal{M}_{B_5}(|\nabla u|^p)^{q_1}\,dx \le N^{q_1}|B_1| + (N^{q_1}-1)|B_1|\sum_{k=1}^{\infty}(\varepsilon_1 N^{q_1})^k$$
$$+\sum_{k=1}^{\infty}\sum_{i=1}^{k}(N^{q_1}-1)N^{q_1 k}\varepsilon_1^i\Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{k-i}\}\Big|.$$

But we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{i=1}^{k}(N^{q_1}-1)N^{q_1 k}\varepsilon_1^i\Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{k-i}\}\Big| \\
&= (\frac{N}{\delta})^{q_1}\sum_{i=1}^{\infty}(\varepsilon_1 N^{q_1})^i\left[\sum_{k=i}^{\infty}(N^{q_1}-1)\delta^{q_1}N^{q_1(k-i-1)}\Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{k-i}\}\Big|\right] \\
&= (\frac{N}{\delta})^{q_1}\sum_{i=1}^{\infty}(\varepsilon_1 N^{q_1})^i\left[\sum_{j=0}^{\infty}(N^{q_1}-1)\delta^{q_1}N^{q_1(j-1)}\Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{j}\}\Big|\right] \\
&\le (\frac{N}{\delta})^{q_1}\Big[\int_{B_1}\mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}})^{q_1}\,dx\Big]\sum_{i=1}^{\infty}(\varepsilon_1 N^{q_1})^i.
\end{aligned}$$

Thus we infer that

$$\begin{aligned}
\int_{B_1}\mathcal{M}_{B_5}(|\nabla u|^p)^{q_1}\,dx &\le N^{q_1}|B_1| + \left[(N^{q_1}-1)|B_1| + (\frac{N}{\delta})^{q_1}\int_{B_1}\mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}})^{q_1}\,dx\right]\sum_{k=1}^{\infty}(\varepsilon_1 N^{q_1})^k \\
&= N^{q_1}|B_1| + \left[(N^{q_1}-1)|B_1| + (\frac{N}{\delta})^{q_1}\int_{B_1}\mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}})^{q_1}\,dx\right]\sum_{k=1}^{\infty}2^{-k} \\
&\le C\left(1 + \int_{B_1}\mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}})^{q_1}\,dx\right)
\end{aligned}$$

with the constant C depending only on p, q, n, Λ, η, $\mathbb{K}$ and M_0. On the other hand, $|\nabla u(x)|^p \le \mathcal{M}_{B_5}(|\nabla u|^p)(x)$ for almost every $x \in B_1$. Therefore, it follows from the strong type $q_1 - q_1$ estimate for the maximal function and the fact $q_1 = q/p$ that

$$\int_{B_1}|\nabla u|^q\,dx \le C\left(1 + \int_{B_5}|\mathbf{F}|^{\frac{q}{p-1}}\,dx\right). \tag{6.11}$$

We next remove the extra assumption (6.7) for u. Notice that for any $M > 0$, by using the weak type $1-1$ estimate for the maximal function and Lemma 3.3 we get

$$\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > NM^p\}\Big| \le \frac{C}{NM^p}\int_{B_5}|\nabla u|^p\,dx \le \frac{C(p,n,\eta,\Lambda)}{M^p}\Big(\int_{B_6}|u|^p\,dx + \int_{B_6}|\mathbf{F}|^{\frac{p}{p-1}}\,dx\Big).$$

Therefore, if we let

$$\overline{u}(x,t) = \frac{u(x,t)}{M} \quad \text{with} \quad M^p = \frac{C(p,n,\eta,\Lambda)\Big[\|u\|^p_{L^p(B_6)} + \||\mathbf{F}|^{\frac{1}{p-1}}\|^p_{L^p(B_6)}\Big]}{\varepsilon|B_1|}$$

then $\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla\overline{u}|^p) > N\}\Big| \le \varepsilon|B_1|$. Hence we can apply (6.11) to $\overline{u}$ with $\mathbf{F}$ and λ being replaced by $\overline{\mathbf{F}} = \mathbf{F}/M^{p-1}$ and $\overline{\lambda} = \lambda M$. By reversing back to the functions u and $\mathbf{F}$, we obtain the desired estimate (2.4). □

We next show that Theorem 1.1 is just a special case of Theorem 2.4.

Proof of Theorem 1.1. For each $0 < \rho \le 3$ and $y \in B_1$, let

$$a_{\rho,y}(z,\xi) := \fint_{B_3} \mathbf{A}(y + \frac{\rho}{3}x, z, \xi)\, dx = \mathbf{A}_{B_\rho(y)}(z,\xi).$$

Then it is easy to see from the assumptions for $\mathbf{A}$ that $a_{\rho,y}$ satisfies (1.2)–(1.4) and (**H1**)–(**H2**) with $\eta := \eta_0 \equiv 0$. Moreover, $a_{\rho,y}$ also satisfies condition (**H3**) thanks to [18, Theorem 1.2]. These imply that $a_{\rho,y} \in \mathbb{G}_{B_3}(\eta_0)$. Thus $a_{\rho,y} \in \mathbb{G}_{B_\rho(y)}(\eta_0)$, and hence

$$\text{dist}\Big(\mathbf{A}, \mathbb{G}_{B_\rho(y)}(\eta_0)\Big) \le \fint_{B_\rho(y)} \Big[\sup_{z\in\overline{\mathbb{K}}} \sup_{\xi\neq 0} \frac{|\mathbf{A}(x,z,\xi) - \mathbf{A}_{B_\rho(y)}(z,\xi)|}{|\xi|^{p-1}}\Big]\, dx \le \delta.$$

Since this holds for every $0 < \rho \le 3$ and $y \in B_1$, Theorem 1.1 follows from Theorem 2.4. □

6.2. **Interior gradient estimates in Orlicz spaces.** In this subsection we show that the interior estimates obtained in Theorem 2.4 still hold true in *Orlicz* spaces. This is achieved by the same arguments as in Subsection 6.1 which illustrates the robustness of our method. This subsection is motivated by [2, Section 4] where Byun and Wang derived similar estimates for the case $\mathbf{A}(x,z,\xi)$ is independent of z.

Let us first recall some basic definitions and properties about *Orlicz* spaces (see [21, 32]). A function $\phi : [0,\infty) \to [0,\infty)$ is called a *Young function* if it is increasing, convex, and

$$\phi(0) = 0, \quad \lim_{t\to 0^+} \frac{\phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

Given a *Young function* ϕ and a bounded domain $U \subset \mathbb{R}^n$, the *Orlicz* space $L^\phi(U)$ is defined to be the linear hull of $K^\phi(U)$ where

$$K^\phi(U) := \Big\{g : U \to \mathbb{R} \text{ measurable } : \int_U \phi(|g|)\, dx < \infty\Big\}.$$

We will need the following well known conditions for ϕ.

Definition 6.4. *Let ϕ be a Young function.*

(i) *ϕ is said to satisfy the $\triangle_2$-condition if there exists a constant $\mu > 1$ such that $\phi(2t) \le \mu\,\phi(t)$ for every $t \ge 0$.*
(ii) *ϕ is said to satisfy the ∇_2-condition if there exists a constant $a > 1$ such that $\phi(t) \le \frac{1}{2a}\phi(at)$ for every $t \ge 0$.*

We will write $\phi \in \triangle_2 \cap \nabla_2$ to mean that ϕ satisfies both (i) and (ii). Notice that $\phi \in \nabla_2$ implies the quasiconvexity of ϕ (see [21, Lemma 1.2.3]).

The next elementary lemma gives a characterization of functions in $L^\phi(U)$ in terms of their distribution functions.

Lemma 6.5. *Assume $\phi \in \triangle_2 \cap \nabla_2$, U is a bounded domain, and $g : U \to \mathbb{R}$ is a nonnegative measurable function. Let $\nu > 0$ and $\alpha > 1$. Then*

$$g \in L^\phi(U) \iff S := \sum_{j=1}^\infty \phi(\alpha^j)\, \Big|\{x \in U : g(x) > \nu\,\alpha^j\}\Big| < \infty.$$

Moreover, there exists $C = C(\nu,\alpha,\phi) > 0$ such that

$$\frac{1}{C} S \le \int_U \phi(g)\, dx \le C(|U| + S).$$

Proof. This follows from the representation formula

$$\int_U \phi(|g|)\,dx = \int_0^\infty \Big|\{x \in U : g(x) > \lambda\}\Big|\,d\phi(\lambda)$$

and the fact $L^\phi(U) \equiv K^\phi(U)$ when $\phi \in \triangle_2$. □

Now we state the version of Theorem 2.4 for *Orlicz* spaces.

Theorem 6.6. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4), *and* $M_0 > 0$. *For any* $q > p$, *there exists a constant* $\delta = \delta(p, q, n, \Lambda, \eta, \mathbb{K}, M_0) > 0$ *such that: if* $\lambda > 0$, $0 < \theta \le 1$,

$$\sup_{0<\rho\le 3}\ \sup_{y\in B_1} dist\Big(\mathbf{A}, \mathbb{G}_{B_\rho(y)}(\eta)\Big) \le \delta,$$

and $u \in W^{1,p}_{loc}(B_6)$ *is a weak solution of* (2.3) *satisfying* $\|u\|_{L^\infty(B_5)} \le \frac{M_0}{\lambda\theta}$, *then:*

$$|\mathbf{F}|^{\frac{p}{p-1}} \in L^\phi(B_5) \quad\Longrightarrow\quad |\nabla u|^p \in L^\phi(B_1). \tag{6.12}$$

In order to prove Theorem 6.6, we need one more lemma concerning about strong type estimates for maximal functions in *Orlicz* spaces.

Lemma 6.7 (Theorem 1.2.1 in [21]). *Assume* $\phi \in \triangle_2 \cap \nabla_2$ *and* $g \in L^\phi(B_5)$. *Then* $\mathcal{M}_{B_5}(g) \in L^\phi(B_5)$ *and*

$$\int_{B_5} \phi(\mathcal{M}_{B_5}(|g|))\,dx \le C(n,\phi) \int_{B_5} \phi(|g|)\,dx.$$

Proof of Theorem 6.6. Since the arguments are essentially the same as those given in Subsection 6.1, we only indicate the main points.

Let $N > 1$ be as in Lemma 6.3. As $\phi \in \triangle_2$, it is easy to see that there exists $\mu > 1$ such that

$$\phi(Nt) \le \mu^{n_0}\phi(t) =: \mu_1\phi(t) \quad \forall t \ge 0, \tag{6.13}$$

where $n_0 \in \mathbb{N}$ depends only on N. Let us choose $\varepsilon = \varepsilon(p, \phi, n, \eta, \Lambda, \mathbb{K}, M_0) > 0$ be such that

$$\varepsilon_1 \overset{\text{def}}{=\!=} 20^n\varepsilon = \frac{1}{2\mu_1},$$

and let $\delta = \delta(p, \phi, n, \Lambda, \eta, \mathbb{K}, M_0)$ be the corresponding constant given by Lemma 6.3. By considering the function $\bar{u} := u/M$ instead of u as done at the end of the proof of Theorem 2.4, we can assume without loss of generality that condition (6.7) is satisfied. Thus, we obtain estimate (6.10) and hence

$$\begin{aligned} &\sum_{k=1}^\infty \phi(N^k)\Big|\{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^k\}\Big| \\ &\le |B_1| \sum_{k=1}^\infty \phi(N^k)\varepsilon_1^k + \sum_{k=1}^\infty \sum_{i=1}^k \phi(N^k)\varepsilon_1^i \Big|\{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{k-i}\}\Big| =: S_1 + S_2. \end{aligned} \tag{6.14}$$

It follows from (6.13) that $\phi(N^k) \le \mu_1^k\phi(1)$ and $\phi(N^k) \le \mu_1^{i-1}\phi(N^{k-i+1})$ for each $i = 1, \ldots, k$. Consequently,

$$S_1 \le \phi(1)|B_1| \sum_{k=1}^\infty (\mu_1\varepsilon_1)^k = \phi(1)|B_1| \sum_{k=1}^\infty 2^{-k} = \phi(1)|B_1| \tag{6.15}$$

and

$$S_2 \le \mu_1^{-1} \sum_{i=1}^{\infty} (\mu_1\varepsilon_1)^i \left[\sum_{k=i}^{\infty} \phi(N^{k-i+1}) \Big| \{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \delta N^{k-i}\} \Big| \right]$$
$$= \mu_1^{-1} \sum_{i=1}^{\infty} (\mu_1\varepsilon_1)^i \left[\sum_{j=1}^{\infty} \phi(N^{j}) \Big| \{B_1 : \mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}}) > \frac{\delta}{N} N^{j}\} \Big| \right].$$

Hence by using Lemma 6.5 and Lemma 6.7, we obtain

$$S_2 \le C \sum_{i=1}^{\infty} (\mu_1\varepsilon_1)^i \int_{B_1} \phi\Big(\mathcal{M}_{B_5}(|\mathbf{F}|^{\frac{p}{p-1}})\Big)\, dx \le C \int_{B_1} \phi\Big(|\mathbf{F}|^{\frac{p}{p-1}}\Big)\, dx.$$

This together with (6.14) and (6.15) yields

$$\sum_{k=1}^{\infty} \phi(N^k) \Big| \{B_1 : \mathcal{M}_{B_5}(|\nabla u|^p) > N^k\} \Big| \le C\Big[1 + \int_{B_1} \phi\Big(|\mathbf{F}|^{\frac{p}{p-1}}\Big)\, dx\Big].$$

Therefore, we conclude from Lemma 6.5 that $\mathcal{M}_{B_5}(|\nabla u|^p) \in L^{\phi}(B_1)$ which gives (6.12) as $|\nabla u(x)|^p \le \mathcal{M}_{B_5}(|\nabla u|^p)(x)$ for a.e. $x \in B_1$. □

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